2. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

for the function $f(z)=e^z$ by

(*a*) using $f^{(n)}(1)$ $(n=0,1,2,\ldots)$; (*b*) writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function

$$f(z)=\frac{z}{z^4+4}=\frac{z}{4}\cdot\frac{1}{1+(z^4/4)}.$$

Ans. $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+2}}z^{4n+1} \quad (|z|<\sqrt{2})$.

4. With the aid of the identity (see Sec. 37)

$$\cos z = -\sin\left(z-\frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0=\pi/2$.

5. Use the identity $\sinh(z+\pi i) = -\sinh z$, verified in Exercise 7(*a*), Sec. 39, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0=\pi i$.

Ans. $-\sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n+1}}{(2n+1)!} \quad (|z-\pi i|<\infty)$.

6. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

7. Show that if $f(z)=\sin z$, then

$$f^{(2n)}(0)=0 \quad\text{and}\quad f^{(2n+1)}(0)=(-1)^n \qquad (n=0,1,2,\ldots).$$

Thus give an alternative derivation of the Maclaurin series (3) for $\sin z$ in Sec. 64.

8. Rederive the Maclaurin series (4) in Sec. 64 for the function $f(z)=\cos z$ by

(*a*) using the definition

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

in Sec. 37 and appealing to the Maclaurin series (2) for e^z in Sec. 64;

(*b*) showing that

$$f^{(2n)}(0)=(-1)^n \quad\text{and}\quad f^{(2n+1)}(0)=0 \qquad (n=0,1,2,\ldots).$$

9. Use representation (3), Sec. 64, for $\sin z$ to write the Maclaurin series for the function

$$f(z)=\sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0)=0 \quad\text{and}\quad f^{(2n+1)}(0)=0 \qquad (n=0,1,2,\ldots).$$

10. Derive the expansions

(*a*) $$\frac{\sinh z}{z^2}=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+3)!} \qquad (0<|z|<\infty);$$

(*b*) $$\frac{\sin(z^2)}{z^4}=\frac{1}{z^2}-\frac{z^2}{3!}+\frac{z^6}{5!}-\frac{z^{10}}{7!}+\cdots \qquad (0<|z|<\infty).$$

11. Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2}=\frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}.$$

66. LAURENT SERIES

We turn now to a statement of ***Laurent's theorem***, which enables us to expand a function $f(z)$ into a series involving positive and negative powers of $(z-z_0)$ when the function fails to be analytic at z_0.

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1<|z-z_0|<R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain* (Fig. 80). *Then, at each point in the domain, $f(z)$ has the series representation*

$$(1) \qquad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} \qquad (R_1<|z-z_0|<R_2),$$

where

$$(2) \qquad a_n=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n=0,1,2,\ldots)$$

and

$$(3) \qquad b_n=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^{-n+1}} \qquad (n=1,2,\ldots).$$

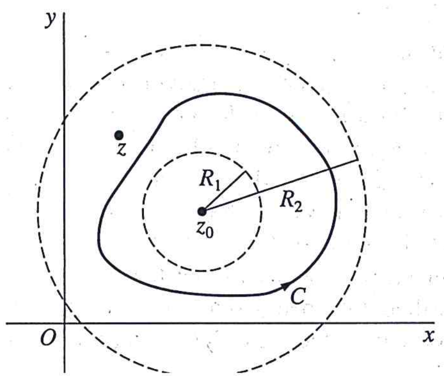

FIGURE 80

Then

$$f(z) = -z\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

Replacing n by $n-1$ in the first of the two series on the far right here yields the desired Maclaurin series:

$$(3) \qquad f(z) = -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -1 - 2\sum_{n=1}^{\infty} z^n \qquad (|z| < 1).$$

The representation of $f(z)$ in the unbounded domain D_2 is a Laurent series, and the fact that $|1/z| < 1$ when z is a point in D_2 suggests that we use series (1) to write

$$f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\frac{1}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$
$$(1 < |z| < \infty).$$

Substituting $n-1$ for n in the last of these series reveals that

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1 < |z| < \infty),$$

and we arrive at the Laurent series

$$(4) \qquad f(z) = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1 < |z| < \infty).$$

EXAMPLE 3. Replacing z by $1/z$ in the Maclaurin series expansion

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we have the Laurent series representation

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \qquad (0 < |z| < \infty).$$

Note that no positive powers of z appear here, since the coefficients of the positive powers are zero. Note, too, that the coefficient of $1/z$ is unity; and, according to Laurent's theorem in Sec. 66, that coefficient is the number

$$b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\,dz$$

where C is any positively oriented simple closed contour around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed contours will be developed in considerable detail in Chap. 6 and then used extensively in Chap. 7.

EXAMPLE 4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n(z-i)^n \qquad (0 < |z-i| < \infty)$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From expression (5), Sec. 66, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$. Thus [compare with Exercise 13, Sec. 46]

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2, \\ 2\pi i & \text{when } n = -2. \end{cases}$$

EXERCISES

1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Ans. $1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}$.

2. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}$.

3. Find the Laurent series that represents the function $f(z)$ in Example 1, Sec. 68, when $1 < |z| < \infty$.

Ans. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$.

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Ans. $\displaystyle\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \qquad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty).$

5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2},$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains (Fig. 84)

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

Ans. $\displaystyle\sum_{n=0}^{\infty}(2^{-n-1} - 1)z^n$ in D_1; $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$ in D_2; $\displaystyle\sum_{n=1}^{\infty}\frac{1 - 2^{n-1}}{z^n}$ in D_3.

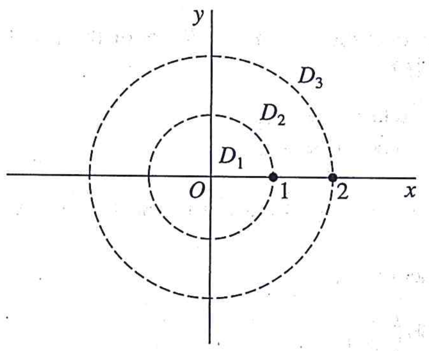

FIGURE 84

6. Show that when $0 < |z - 1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. (*a*) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad (|a| < |z| < \infty).$$

(*b*) After writing $z = e^{i\theta}$ in the equation obtained in part (*a*), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 61.)

8. Suppose that a series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. That sum $X(z)$ is called the ***z*-transform** of $x[n]$ $(n = 0, \pm1, \pm2, \ldots)$.* Use expression (5), Sec. 66, for the coefficients in a Laurent series to show that if the annulus contains the unit circle $|z| = 1$, then the *inverse* z-transform of $X(z)$ can be written

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta})e^{in\theta}\,d\theta \qquad (n = 0, \pm1, \pm2, \ldots).$$

9. (*a*) Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \qquad (-\pi \le \phi \le \pi)$$

in the w plane. Then use that contour in expression (5), Sec. 66, for the coefficients in a Laurent series, adapted to such series about the origin in the w plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \qquad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-i(n\phi - z\sin\phi)]\,d\phi \qquad (n = 0, \pm1, \pm2, \ldots).$$

(*b*) With the aid of Exercise 5, Sec. 42, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part (*a*) here can be written†

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\phi - z\sin\phi)\,d\phi \qquad (n = 0, \pm1, \pm2, \ldots).$$

10. (*a*) Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 66, for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

when z is any point in the annular domain.

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*The z-transform arises in studies of discrete-time linear systems. See, for instance, the book by Oppenheim, Schafer, and Buck that is listed in Appendix 1.

†These coefficients $J_n(z)$ are called ***Bessel functions*** of the first kind. They play a prominent role in certain areas of applied mathematics. See, for example, the authors' "*Fourier Series and Boundary Value Problems*," 8th ed., Chap. 9, 2012.