THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Test 2, Nov 10, 2016
SAMPLE SOLUTIONS

Name:____________________________ID:_____________________ Marks:__________

Number of questions: 5. Full marks: 50
Answer all questions, show your work!

1. (20 pts)

   (a) Let $f(x,y) = \dfrac{y^2}{x^3+y}$. Find $\dfrac{\partial f}{\partial y}$.

   $$\frac{\partial f}{\partial y} = \frac{(x^3+y)2y - y^2}{(x^3+y)^2}$$

   (b) Let $f(x,y) = \cos(x^2y + \ln y)$. Find $\dfrac{\partial^2 f}{\partial y \partial x}$.

   $$\frac{\partial f}{\partial x} = -2xy\sin(x^2y + \ln y).$$

   Hence,

   $$\frac{\partial f}{\partial y \partial x} = -2xy\cos(x^2y + \ln y)\left(x^2 + \frac{1}{y}\right) - 2x\sin(x^2y + \ln y).$$

(c) Let $f(x,y,z) = (xz)^y$ for $x, z > 0$. Find $\dfrac{\partial^2 f}{\partial x \partial z}$ and $\dfrac{\partial^2 f}{\partial y \partial z}$.

$$\frac{\partial f}{\partial z} = y(xz)^{y-1}x = yx^y z^{y-1}.$$

Hence,

$$\frac{\partial^2 f}{\partial x \partial z} = y^2(xz)^{y-1}.$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial z} &= \frac{\partial}{\partial y}\left(y(xz)^{y-1}x\right) \\ &= \frac{\partial}{\partial y}\left(y(xz)^y \cdot \frac{1}{xz}x\right) \\ &= \frac{\partial}{\partial y}\left(\frac{1}{z} \cdot y(xz)^y\right) \\ &= \frac{1}{z}\left((xz)^y + y\ln(xz)(xz)^y\right)\end{aligned}$$

2. (5 pts) Let $\mathcal{P}$ be the plane in xyz-space which contains the line:

$$\vec{l_1}(t) = \langle -1, 0, 5\rangle t + \langle 5, -6, 7\rangle, \quad t \in \mathbb{R};$$

and is parallel to the line:

$$\vec{l_2}(t) = \langle 2, 1, 3\rangle t + \langle 0, 8, 23\rangle, \quad t \in \mathbb{R}.$$

Find an equation in x, y, z whose graph is the plane $\mathcal{P}$.

Any normal vector $\vec{n}$ of the plane is perpendicular to both $\langle -1, 0, 5\rangle$ and $\langle 2, 1, 3\rangle$.

So, we may take:

$$\vec{n} = \langle -1, 0, 5\rangle \times \langle 2, 1, 3\rangle = \langle -5, 13, -1\rangle$$

Given that the plane contains the line $\vec{l_1}$. We know that the point $(5, -6, 7)$ lies on the plane. We conclude that the plane is the graph of the equation:

$$-5(x-5) + 13(y+6) - (z-7) = 0.$$

3. (8 pts) Evaluate each of the following limits, if it exists. If it does not exist, explain why not.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$

   The limit of $\dfrac{x^2+y^2}{x^2-y^2}$ as (x,y) approaches $(0,0)$ along the path $(x(t),y(t))=(t,0)$ is:

   $$\lim_{t\to 0} \frac{t^2}{t^2} = 1.$$

   The limit of $\dfrac{x^2+y^2}{x^2-y^2}$ as (x,y) approaches $(0,0)$ along the path $(x(t),y(t))=(0,t)$ is:

   $$\lim_{t\to 0} \frac{t^2}{-t^2} = -1.$$

   Since the limits along two different paths are not equal to each other, the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$ **does not exist**.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

   Since $x^2+y^2 \geq x^2$, we have:

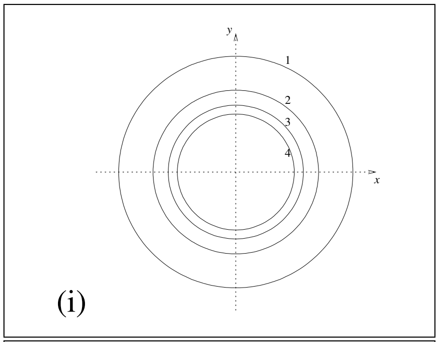
   $$-|x| = -\left|\frac{x^3}{x^2}\right| \leq \frac{x^3}{x^2+y^2} \leq \left|\frac{x^3}{x^2}\right| = |x|\,.$$

   Since:

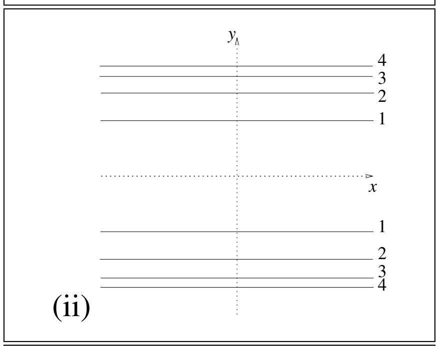
   $$\lim_{(x,y)\to(0,0)} -|x| = \lim_{(x,y)\to(0,0)} |x| = 0,$$

   by Sandwich Thereom we conclude that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$

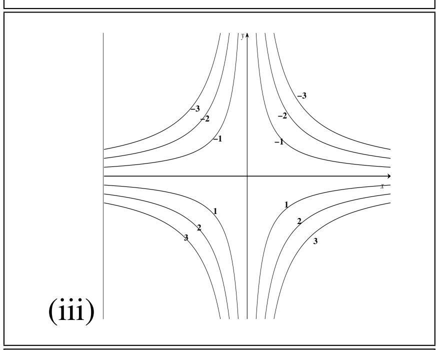
4. (10 pts) For each of the level set diagrams on the left, indicate which function or graph on the right it might possibly correspond to. If there is no possible match, write “none”.



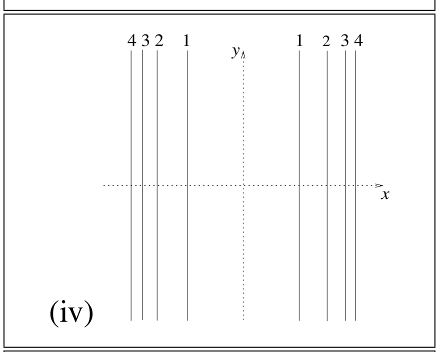
(i) H



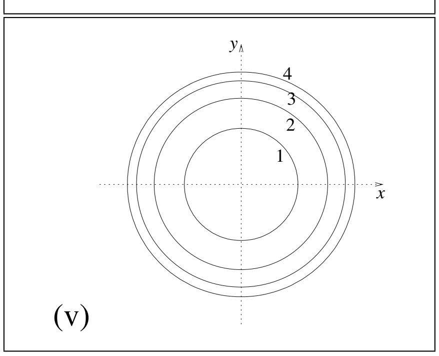
(ii) D



(iii) None



(iv) None



(v) F

(A) $f(x,y) = xy$

(B) $f(x,y) = x - y$

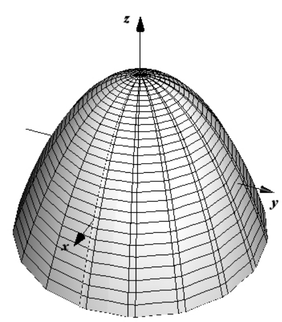
(C) $f(x,y) = x + y$

(D) $f(x,y) = y^2$

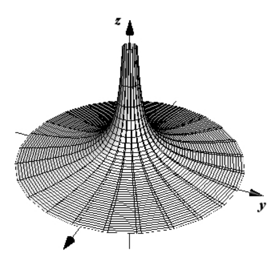
(E) $f(x,y) = x^2 + 1$

(F) $f(x,y) = x^2 + y^2$

(G)



(H)



5. (7 pts) Let:

$$f(x,y) = \begin{cases} \frac{x^2y - xy^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

Evaluate each of the following partial derivatives, or show that it does not exist.

(a) $f_x(0,0)$

(b) $f_{xy}(0,0)$

(a) By definition,

$$\begin{aligned} f_x(0,0) &= \frac{d}{dx} f(x,0)\bigg|_{x=0} \\ &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^2\cdot 0 - h\cdot 0^2}{h^2+0^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= 0 \end{aligned}$$

(b) By definition,

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

We have shown that $f_x(0,0) = 0$.

For $h \neq 0$,

$$\begin{aligned} f_x(0,h) &= \lim_{t\to 0} \frac{f(t,h) - f(0,h)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^2h - th^2}{t^2+h^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{th - h^2}{t^2 + h^2} \\ &= -1 \end{aligned}$$

Hence,

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{-1-0}{h} = \lim_{h\to 0} \frac{-1}{h},$$

which does not exist.

End of Paper