THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Test 1, Oct 13, 2016
Time allowed: 45 mins

Name:____________________________ID:____________________ Marks:__________

Number of questions: 5. Full marks: 50
Answer all questions, show your work!

1. (15 pts) Find all solutions $\vec{x}$ to the following matrix equations:

   (a)

$$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad \vec{x} \in \mathbb{R}^3$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & -1 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -7 \\ 0 & 1 & 0 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 5 & 22 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 5 & 25 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

So,

$$x_3 = 5$$
$$x_2 = 3$$
$$x_1 - x_3 = -7$$

We conclude that:

$$\vec{x} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} -2 & -1 & 0 & 4 \\ 1 & 0 & 0 & -5 \\ -1 & -1 & 1 & 9 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \vec{x} \in \mathbb{R}^4$$

$$\left(\begin{array}{cccc|c} -2 & -1 & 0 & 4 & 1 \\ 1 & 0 & 0 & -5 & -1 \\ -1 & -1 & 1 & 9 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -1 \\ -2 & -1 & 0 & 4 & 1 \\ -1 & -1 & 1 & 9 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -1 \\ 0 & -1 & 0 & -6 & -1 \\ 0 & -1 & 1 & 4 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -1 \\ 0 & -1 & 0 & -6 & -1 \\ 0 & 0 & 1 & 10 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -1 \\ 0 & 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & 10 & 3 \end{array}\right)$$

This implies that:

$$\begin{aligned} x_3 + 10x_4 &= 3 \\ x_2 + 6x_4 &= 1 \\ x_1 - 5x_4 &= -1 \end{aligned}$$

The unknown x_4 can be any real number t, hence we have:

$$\begin{aligned} x_3 &= 3 - 10t \\ x_2 &= 1 - 6t \\ x_1 &= -1 + 5t. \end{aligned}$$

The solutions to the matrix equation are therefore:

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ -6 \\ -10 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

2. (10 pts) Find all $x \in \mathbb{R}$ for which the following matrix is invertible:

$$M = \begin{pmatrix} x^2 & 3x & x \\ -2 & -x & 0 \\ 4 & 5 & 0 \end{pmatrix}.$$

The matrix M is invertible *if and only* if $\det M \neq 0$. By direct computation we have $\det M = x(10 + 4x)$, hence the matrix M is invertible if and only if x is not equal to 0 or $5/2$.

3. (8 pts) Find a vector parameterization (in the form $\vec{l}(t) = t\vec{v} + \vec{v}_0$, $t \in \mathbb{R}$) for a line in $\mathbb{R}^3$ which is perpendicular to the plane:

$$x - y + z = 256,$$

and contains the point $(1, 1, 3)$. If such a line does not exist, explain why not.

A normal vector to the plane is $\vec{n} = \langle 1, -1, 1 \rangle$. By hypothesis the line is parallel to $\vec{n}$, hence a vector parameterization for the line is:

$$\vec{l}(t) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}.$$

4. (10 pts) Determine if each of the following statements is **true** or **false**. You are not required to justify your answers *for this problem*.
For each statement, you get 2 points for answering correctly, 1 point for leaving it blank, and 0 point for answering incorrectly.

(a) For any $n \times n$ matrix A, if $A^l = \underbrace{AA\cdots A}_{l \text{ times}}$ is non-invertible for some $l \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then A is non-invertible.
True. $\det A^l = (\det A)^l = 0$ imples that $\det A = 0$, which implies that A is non-invertible.

(b) For all linear maps $\mathcal{C}, \mathcal{D}$ from $\mathbb{R}^n$ to $\mathbb{R}$, their sum $\mathcal{C} + \mathcal{D} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is necessarily linear. (By definition: $(\mathcal{C} + \mathcal{D})(\vec{v}) = \mathcal{C}(\vec{v}) + \mathcal{D}(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$.)
True. For any $\vec{v}, \vec{w} \in \mathbb{R}^n$, we have:

$$(\mathcal{C} + \mathcal{D})(\vec{v} + \vec{w}) = \mathcal{C}(\vec{v} + \vec{w}) + \mathcal{D}(\vec{v} + \vec{w}),$$

which by the linearity of $\mathcal{C}$ and $\mathcal{D}$ is equal to:

$$\mathcal{C}(\vec{v})+\mathcal{C}(\vec{w})+\mathcal{D}(\vec{v})+\mathcal{D}(\vec{w}) = \mathcal{C}(\vec{v})+\mathcal{D}(\vec{v})+\mathcal{C}(\vec{w})+\mathcal{D}(\vec{w}) = (\mathcal{C}+\mathcal{D})(\vec{v})+(\mathcal{C}+\mathcal{D})(\vec{w}).$$

For any $\lambda \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, we have:

$$(\mathcal{C} + \mathcal{D})(\lambda\vec{v}) = \mathcal{C}(\lambda\vec{v}) + \mathcal{D}(\lambda\vec{v}) = \lambda\mathcal{C}(\vec{v}) + \lambda\mathcal{D}(\vec{v}) = \lambda(\mathcal{C} + \mathcal{D})(\vec{v}).$$

Hence, the map $\mathcal{C} + \mathcal{D}$ is linear.

(c) For any $n \times n$ matrix A and $n \times n$ invertible matrix B, the equation $(B^{-1}AB)\vec{x} = \vec{0}$ has a unique solution $\vec{x}$ if and only if $A\vec{x} = \vec{0}$ has a unique solution.
True. For a square matrix M, the equation $M\vec{x} = \vec{0}$ has a unique solution if and only if $\det M$ is nonzero.
Since $\det(B^{-1}AB) = (\det B)^{-1}(\det A)(\det B) = \det A$, the equation $(B^{-1}AB)\vec{x} = \vec{0}$ has a unique solution if and only if $A\vec{x} = \vec{0}$ has a unique solution.

(d) Two unit vectors are parallel to each other if and only if their dot product is nonzero.
False. Take $\vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$, $\vec{w} = \langle 1, 0\rangle$. These unit vectors are non-parallel, but $\vec{v} \cdot \vec{w} = 1/\sqrt{2} \neq 0$.

(e) Any $n \times n$ matrix is row equivalent to some matrix A which has the property: $A_{ij} = 0$ if $i > j$.
True. By a theorem shown in class, any square matrix is row equivalent to an upper triangular matrix.

5. (7 pts) Let $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{w} = \langle w_1, w_2, w_3\rangle$ be nonzero vectors in $\mathbb{R}^3$ such that:

$$\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} = 0\,.$$

Show that the matrix $A = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$ is invertible.

Suppose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a solution to $A\vec{x} = \vec{0}$. Then, we have:

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0} \qquad (*)$$

Taking the dot products of both sides of the equation above with $\vec{u}$, we have:

$$x_1\,|\vec{u}|^2 = 0,$$

since $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w} = 0$ by hypothesis.

Since $\vec{u} \neq \vec{0}$, we have $|\vec{u}|^2 \neq 0$, which implies that $x_1 = 0$.

By taking the dot products of both sides of $(*)$ with $\vec{v}$ and then with $\vec{w}$, we conclude by the same argument that x_2 and x_3 are also equal to 0.

Hence, $\vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$. So, by a theorem shown in class the matrix A is invertible.

End of Paper