Double Integrals over More General Regions

Let:

$$R = \{(x, y) \in \mathbb{R}^2 \ : \ a \leq x \leq b, c(x) \leq y \leq d(x)\},$$

where $c(x), d(x)$ are continuous functions in x. Then:

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$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \\ &= \int_{x=a}^{x=b} \left[\int_{y=c(x)}^{y=d(x)} f(x,y)\,dy\right] dx \\ &= \int_{x=a}^{x=b} [F(x,c(x)) - F(x,d(x))]\,dx,\end{aligned}$$

where $F(x, y)$ is a function in two variables such that $\dfrac{\partial F}{\partial y} = f(x, y)$.

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Similarly, if:

$$R = \{(x, y) \in \mathbb{R}^2 \ : \ c \leq y \leq d, a(y) \leq x \leq b(y)\},$$

where $a(y), b(y)$ are continuous functions in y, then:

$$\begin{aligned}\iint_R f(x,y)dA &= \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy \\ &= \int_{y=c}^{y=d} \left[\int_{x=a(y)}^{x=b(y)} f(x,y)\,dx\right] dy \\ &= \int_{y=c}^{y=d} [G(a(y),y) - G(b(y),y)]\,dy,\end{aligned}$$

where $G(x, y)$ is a function in two variables such that $\dfrac{\partial G}{\partial x} = f(x, y)$.

Example.

Evaluate:

- $$\int_0^4 \int_{\sqrt{x}}^{x} (2xy + y)e^{x+y^2}\,dy\,dx.$$
- $$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx.$$

For a bounded closed region $R \subseteq \mathbb{R}^2$, the area of R is equal to:

$$\iint_R 1\, dA.$$

(i.e. the double integral of the constant function $f(x, y) = 1$ over R).

Example.

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Find the area of region R in $\mathbb{R}^2$ bounded by the curves $y = x - 1$, $y = \sqrt{x}$, and the x-axis.

Example.

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Find the area of region R in $\mathbb{R}^2$ bounded by the curves $y = x$ and $y = x^3$.

Triple Integrals

Consider the solid $D \subseteq \mathbb{R}^3$ defined as follows:

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \begin{array}{ccccc} a & \leq & x & \leq & b \\ y_1(x) & \leq & y & \leq & y_2(x) \\ z_1(x, y) & \leq & z & \leq & z_2(x, y) \end{array} \right\},$$

where $y_1(x)$, $y_2(x)$ are continuous functions in x, and $z_1(x, y)$, $z_2(x, y)$ are continuous functions in (x, y).

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In other words, the solid is bounded from above by the surface $z = z_2(x, y)$, and from below by the surface $z = z_1(x, y)$. Along the direction parallel to the y-axis, the solid is bounded by the vertical surfaces $y = y_1(x)$ and $y = y_2(x)$. Along the direction parallel to the x-axis, the solid is bounded between the vertical planes $x = a$ and $x = b$.

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The **triple integral** $\iiint_D f(x, y, z)\, dV$ of a continuous function $f(x, y, z)$ over D is equal to:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) dz \right] dy \right] dx.$$

There are solids in $\mathbb{R}^3$ defined similarly, but with the conditions on x, y and z permuted. For example, we can have:

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \begin{array}{ccccc} c & \leq & y & \leq & d \\ z_1(y) & \leq & z & \leq & z_2(y) \\ x_1(y, z) & \leq & x & \leq & x_2(y, z) \end{array} \right\}.$$

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Then,

$$\iiint_D f(x,y,z)\,dV = \int_c^d \int_{z_1(y)}^{z_2(y)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Example.
Evaluate:

- $$\int_0^2 \int_1^2 \int_0^1 yze^{xz}\,dz\,dy\,dx$$
- $$\int_0^1 \int_0^z \int_{z+y}^{y^2} \sqrt{x}\,dx\,dy\,dz$$

Evaluate:

$$\int_0^2 \int_1^2 \int_0^1 yze^{xz}\,dz\,dy\,dx$$

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After a change of order of integration, we have:

$$\int_0^2 \int_1^2 \int_0^1 yze^{xz}\,dz\,dy\,dx = \int_0^1 \int_1^2 \int_0^2 yze^{xz}\,dx\,dy\,dz,$$

which is equal to:
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$$\begin{aligned}
\int_0^1 \int_1^2 ye^{xz}\Big|_{x=0}^{x=2} dy\,dz &= \int_0^1 \int_1^2 y\left(e^{2z}-1\right)\,dy\,dz \\
&= \int_0^1 \frac{1}{2}\left(e^{2z}-1\right) y^2\Big|_{y=1}^{y=2} dz \\
&= \int_0^1 \frac{3}{2}\left(e^{2z}-1\right)\,dz \\
&= \frac{3}{2}\left(\frac{1}{2}e^{2z}-z\right)\Big|_{z=0}^{z=1} \\
&= \frac{3}{2}\left(\frac{1}{2}\left(e^2-1\right)-1\right) = \frac{3}{4}e^2 - \frac{9}{4}
\end{aligned}$$

The volume of a closed and bounded solid $D \subseteq \mathbb{R}^3$ is equal to:

$$\iiint_D 1\,dV$$

Example.

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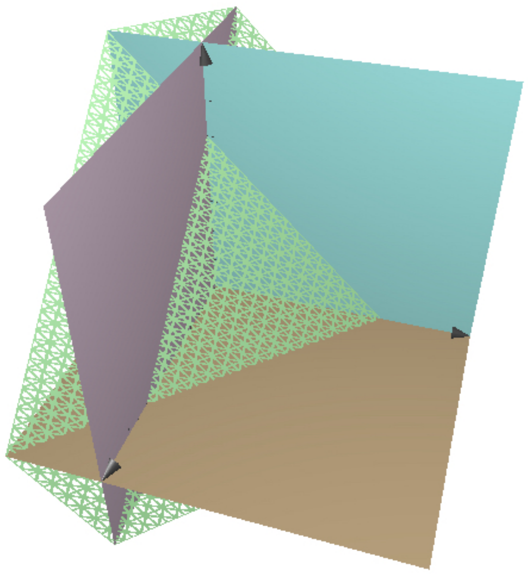
Find the volume of:

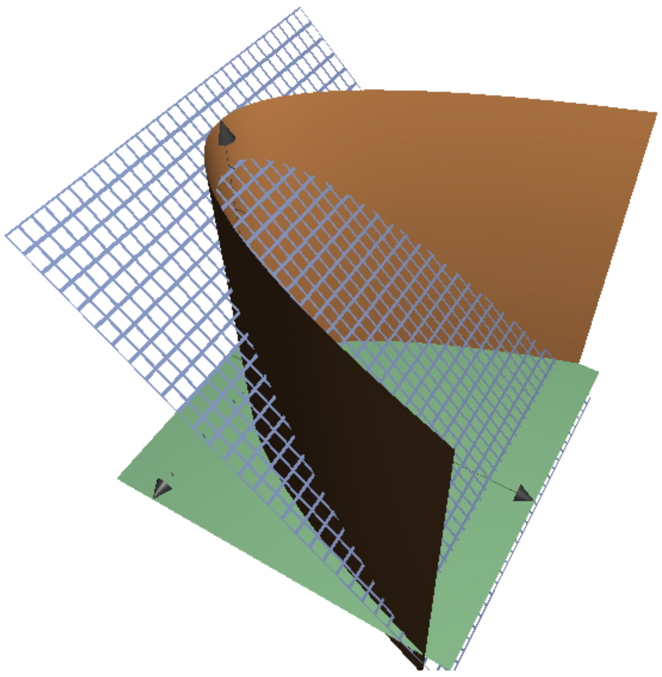
- The solid in the first octant $(x, y, z \geq 0)$ of $\mathbb{R}^3$ bounded by: the plane $x + y + z = 1$ and the xy-, xz- and yz-planes.
  Solution.



Evaluate:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx.$$

- The solid in $\mathbb{R}^3$ bounded by: the cylinder $y = x^2$, the plane $z = 3 - y$, and the xy-plane.
  Solution.



Evaluate:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_0^{3-y} dz\, dy\, dx.$$