Week 12
Taylor's Theorem
Local Extrema

Taylor's Theorem for Functions in Two Variables

Let $f(x, y)$ be a function in two variables, $n \in \mathbb{N}$. Suppose the partial derivatives of f of all orders up to $n+1$ exist and are continuous at all points in an open ball B of positive radius centred at (a, b), then for $(x, y) \in B$, we have:

$$f(x, y) = p_n(x, y) + +R_n(x, y),$$

where:

$$p_n(x, y) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^k f}{\partial x^{k-j} \partial y^j}\bigg|_{(a,b)} (x-a)^{k-j}(y-b)^j$$

$$= f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$
$$+ \frac{1}{2!}\left(f_{xx}(a, b)(x-a)^2 + 2f_{xy}(a, b)(x-a)(y-b) + f_{yy}(a, b)(y-b)^2\right)$$
$$+ \frac{1}{3!}\left(f_{xxx}(a, b)(x-a)^3 + 3f_{xxy}(a, b)(x-a)^2(y-b)\right.$$
$$\left.+3f_{xyy}(a, b)(x-a)(y-b)^2 + f_{yyy}(a, b)(y-b)^3\right) + \cdots,$$

and:

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$$R_n(x, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{\partial^{n+1} f}{\partial x^{n+1-j} \partial y^j}\bigg|_{(a+c(x-a), b+c(x-b))} (x-a)^{n+1-j}(y-b)^j,$$

for some $c \in (0, 1)$.
The polynomial $p_n(x, y)$ is called the **n-th Taylor Polynomial** of $f(x, y)$ about (a, b).

Example.

Let $f(x,y)=\sin x\sin y$. Approximate the value of $f(0.01,-0.2)$ using the second Taylor Polynomial of f about $(0,0)$.
We have:

$$f_x(x,y)=\cos x\sin y,\quad f_y(x,y)=\sin x\cos y,$$
$$f_{xx}(x,y)=-\sin x\sin y,\quad f_{xy}(x,y)=\cos x\cos y,\quad f_{yy}(x,y)=-\sin x\sin y.$$

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Hence, the second Taylor Polynomial of f about $(0,0)$ is:

$$\begin{aligned}p(x,y)&=f(0,0)+f_x(0,0)x+f_y(0,0)y\\&\quad+\frac{1}{2!}\left(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2\right)\\&=0+0+0+\frac{1}{2!}(0+2\cdot 1\cdot xy+0)=xy.\end{aligned}$$

So, $f(0.01,-0.2)$ is approximately equal to $p(0.01,-0.2)=(0.01)(-0.2)=-0.002$.

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The error of the approximation is:

$$\begin{aligned}|f(0.01,-0.2)-p(0.01,-0.2)|&=|R_2(0.01,-0.2)|\\&=\Big|\frac{1}{3!}\big(f_{xxx}(0.01c,-0.2c)(0.01)^3+3f_{xxy}(0.01c,-0.2c)(0.01)^2(-0.2)\\&\quad+3f_{xyy}(0.01c,-0.2c)(0.01)(-0.2)^2+f_{yyy}(0.01c,-0.2c)(-0.2)^3\big)\Big|,\end{aligned}$$

for some $c\in(0,1)$.

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Computing the 3-rd order partial derivatives of f, we have:

$$\begin{aligned}&|R_2(0.01,-0.2)|\\&=\Big|\frac{1}{3!}\big(-\cos(0.01c)\sin(-0.2c)(0.01)^3-3\sin(-0.01c)\cos(-0.2c)(0.01)^2(-0.2)\\&\quad-3\cos(0.01c)\sin(-0.2c)(0.01)(-0.2)^2-\sin(0.01c)\cos(-0.2c)(-0.2)^3\big)\Big|\\&\le\frac{1}{6}\left(|0.01|^3+3|0.01|^2|-0.2|+3|0.01||-0.2|^2+|-0.2|^3\right),\end{aligned}$$

since the sine and cosine functions have absolute values less than or equal to 1.

Example.

Find the 3rd Taylor polynomial of $f(x,y)=\ln(2x+y)$ at the point $(0,1)$.

In general, for a function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ in n variables, its l-th Taylor polynomial at a point $\vec{a}=(a_1,a_2,\ldots,a_n)$ is:

$$p_n(\vec{x}) = \sum_{k=0}^{l} \frac{1}{k!} \sum_{j_1+j_2+\ldots+j_n=k} \underbrace{\frac{k!}{j_1!j_2!\cdots j_n!}}_{\text{From the ``Multinomial Theorem''.}} \left.\frac{\partial^k f}{\partial^{j_1}x_1\partial^{j_2}x_2\cdots\partial^{j_n}x_n}\right|_{\vec{x}=\vec{a}} (x_1-a_1)^{j_1}(x_2-a_2)^{j_2}\cdots(x_n-a_n)^{j_n}$$

$$= \sum_{k=0}^{l} \sum_{j_1+j_2+\ldots+j_n=k} \frac{1}{j_1!j_2!\cdots j_n!} \left.\frac{\partial^k f}{\partial^{j_1}x_1\partial^{j_2}x_2\cdots\partial^{j_n}x_n}\right|_{\vec{x}=\vec{a}} (x_1-a_1)^{j_1}(x_2-a_2)^{j_2}\cdots(x_n-a_n)^{j_n}$$

Local Extrema

We say that a function f in two variables has a **local minimum** (resp. **local maximum**) at (a, b) if there exists an open disk D of positive radius, centred at (a, b), such that $f(a, b) \leq f(x, y)$ (resp. $f(a, b) \geq f(x, y)$) for all $(x, y) \in D$.

Definition.

Let f be a function defined on a region D in $\mathbb{R}^2$. We say that an interior point $(a, b) \in D$ is a **critical point** of f if $\nabla f(a, b)$ is either equal to $\langle 0, 0\rangle$ or underfined (i.e. one or both of $f_x(a, b)$, $f_y(a, b)$ does not exist.)

Definition.

We say that $f(x, y)$ has a **saddle point** at a critical point (a, b) if for all open disks D of positive radius centred at (a, b), there exists $(x_1, y_1) \in D$ such that $f(a, b) \leq f(x_1, y_1)$, and there exists $(x_2, y_2) \in D$ such that $f(a, b) \geq f(x_2, y_2)$.

Theorem.

If a function f defined on a region $D \subseteq \mathbb{R}^2$ has a local extremum (i.e. local max or min) at $(a, b) \in D$, then (a, b) is either a critical point of f or a boundary point of D.

Second Deriviative Test

Let $f(x, y)$ be a function in two variables (with continuous second order partial derivatives). Define:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 \quad \left(= \det \underbrace{\begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}}_{\text{``Hessian'' matrix}} \right)$$

Theorem.

(**Second Derivative Test**) Suppose (a, b) is a critical point of f, and the first and second order partial derivatives of f are continuous on an open neighborhood of (a, b) (in particular $\nabla f(a, b) = \vec{0}$). Then:

- If $D(a, b) > 0$:
  - If $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
  - If $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D(a, b) < 0$:
  f has a saddle point at (a, b).

If $D(a, b) = 0$, The second derivative test is inconclusive.

Example.

Let:

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

Classify the critical points of f.

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$$\nabla f(x, y) = \langle -6x + 6y, 6y - 6y^2 + 6x \rangle,$$

which is defined for all (x, y).

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Solving:

$$\nabla f(x, y) = \langle 0, 0 \rangle,$$

We obtain:

$$(x, y) = (0, 0) \text{ or } (2, 2).$$

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$$f_{xx} = -6, \quad f_{xy} = 6, \quad f_{yy} = 6 - 12y.$$

Hence,

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 72(y - 1).$$

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Evaluating $D(x, y)$ at the critical points, we have:

$$D(0, 0) = -72 < 0.$$
$$D(2, 2) = 72 > 0.$$

This implies that:

$(0, 0)$ corresponds to a saddle point,

and that $(2, 2)$ corresponds to either a local maximum or minimum.

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Since, $f_{xx}(2,2) = -6 < 0$, we conclude that:

$$\boxed{(2,2) \text{ corresponds to a local maximum.}}$$

Idea Behind the Second Derivative Test

Let (a,b) be the critical point under consideration. By Taylor's Theorem, over a small neighborhood of (a,b), $f(x,y)$ is closely approximated by the polynomial:

$$\begin{aligned} Q(x,y) = f(a,b) + \underbrace{f_x(a,b)}_{=0}(x-a) + \underbrace{f_y(a,b)}_{=0}(y-b) \\ + \frac{1}{2}\left(f_x(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right). \end{aligned}$$

The polynomial Q is of degree 2, and the graphs of such polynomials fall into 3 categories:

- Downward paraboloid. This corresponds to $D(a,b) > 0$, $f_{xx}(a,b) < 0$.
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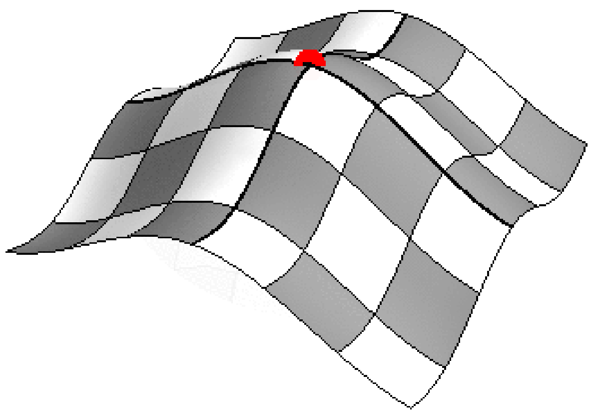

- Upward paraboloid. This corresponds to $D(a,b) > 0$, $f_{xx}(a,b) > 0$.
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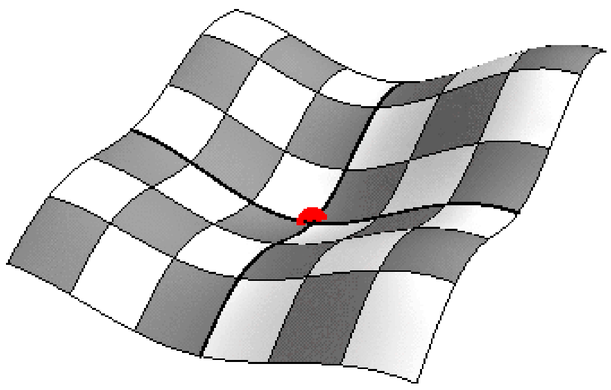

- Hyperbolic paraboloid. This corresponds to $D(a,b) < 0$.
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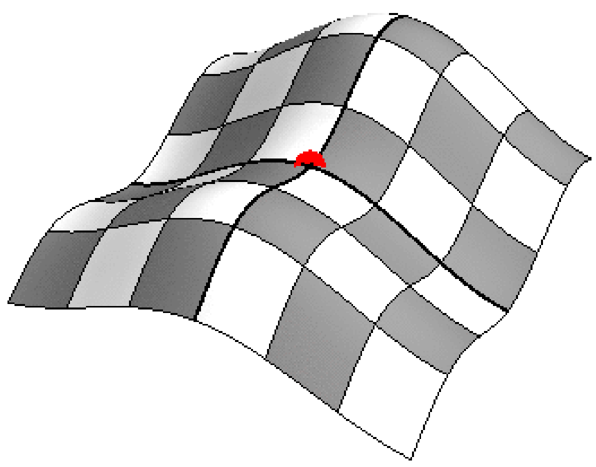

From the pictures one can see that the three cases correspond to local maximum, minimum, and saddles points, respectively.

(Illustration by Blacklemon67 - made with mathematica, CC BY-SA 3.0, Link.)

Multiple Integrals

Double Integrals over Rectangular Regions

Let $f(x,y)$ be a continuous function on a rectangular region:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \ : \ a \le x \le b, c \le y \le d\}$$

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Partition the interval $[a,b]$ into m subintervals of equal length $\Delta x = \dfrac{b-a}{m}$,

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and likewise partition $[c,d]$ into n subintervals of equal length $\Delta y = \dfrac{d-c}{n}$.

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Definition.

Given that f is continuous, the **double integral** $\iint_R f(x,y)\,dA$ of f over R is the limit as $n, m \to \infty$ of the double Riemann sum:

$$\sum_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} f(x_i, y_j) \Delta x \Delta y,$$

where:

$$x_i = a + i\Delta x, \quad y_j = c + j\Delta y.$$

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http://www2.stetson.edu/~wmiles/coursedocs/Fall_05/MS_203/calc3labs/Calculus%20III%20-%20Lab%209.htm

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Definition.

The integrals:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx := \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy \right] dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy := \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx \right] dy$$

are called **iterated integrals** of f over $R = [a,b] \times [c,d]$.

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Here, $\int_c^d f(x,y)\,dy$ should be viewed as the integral of a one-variable function $f(x,y)$ in y, with x fixed. In other words:

$$\int_{y=c}^{y=d} f(x,y)\,dy = F(x,y)\Big|_{y=c}^{y=d} = F(x,c) - F(x,d),$$

where $F(x,y)$ is a function in two variables such that $\dfrac{\partial F}{\partial y} = f(x,y)$.

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Hence,

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b [F(x,c) - F(x,d)]\,dx,$$

which is an integral of a one-variable function in x.

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Likewise,

$$\int_{x=a}^{x=b} f(x,y)\,dx = G(x,y)\Big|_{x=a}^{x=b} = G(a,y) - G(b,y),$$

where $\dfrac{\partial G}{\partial x} = f(x,y)$, and:

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$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d [G(a,y) - G(b,y)]\,dy,$$

which is an integral of a one-variable function in y.

Theorem.

(**Fubini's Theorem**) If $f(x,y)$ is continuous over $R = [a,b] \times [c,d]$, then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example.

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Compute:

- $$\int_0^1 \int_2^4 3x^2 y\,dy\,dx$$
- $$\int_{-1}^1 \int_2^3 xye^x\,dx\,dy$$
- $$\iint_{[0,1]\times[0,2]} \frac{xy^2}{(x^2+y)^2}\,dA$$
