Math 2230A, Complex Variables with Applications

1. Show that

$$(a)\exp(2\pm 3\pi i) = -e^2; \quad (b)\exp\left(\tfrac{2+\pi i}{4}\right) = \sqrt{\tfrac{e}{2}}(1+i)$$
$$(c)\exp(z+\pi i) = -\exp z$$

2. Write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$\left|\exp(2z+i) + \exp\left(iz^2\right)\right| \le e^{2x} + e^{-2xy}.$$

3. Show that $|\exp(-2z^2)| \le \exp(|z|^2)$.

4. Prove that $|\exp(-2z)| < 1$ if and only if $Re z > 0$.

5. Find all values of z such that
   (a) $e^z = -2$;
   (b) $e^z = 1+i$;
   (c) $\exp(2z-1) = 1$.

6. Show that
   (a) $Log(-ei) = 1 - \frac{\pi}{2}i$; (b) $Log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

7. Show that
   (a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
   (b) $\log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
   (c) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

8. Show that $Log(i^3) \ne 3Log i$.

9. Show that $\log(i^2) \ne 2\log i$ when the branch

$$\log z = lnr + i\theta \quad (r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$$

   is used.

10. (a) Show that the two square roots of i are

$$e^{i\pi/4} \quad \text{and} \quad e^{i5\pi/4}.$$

   Then show that

$$\log(e^{i\pi/4}) = \left(2n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \left[(2n+1) + \frac{1}{4}\right]\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Conclude that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Show that

$$\log(i^{1/2}) = \frac{1}{2}\log i,$$

as stated in Example 5, Sec. 32, by finding the values on the right-hand side of this equation and then comparing them with the final result in part (a).

11. Find all roots of the equation $\log z = i\pi/2$.

12. Suppose that the point $x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = lnr + i\theta(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$.