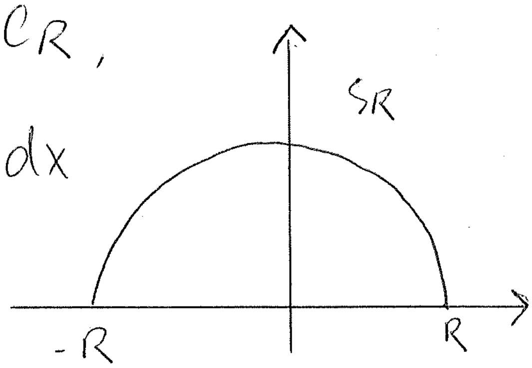
① Prove $\int_{-\infty}^{\infty} \frac{1}{1+x^6} = \frac{2\pi}{3}$.

Ans: We consider the path C_R,

$$\int_{C_R} \frac{1}{1+z^6}\,dz = \int_{S_R} \frac{1}{1+z^6}\,dz + \int_{-R}^{R} \frac{1}{1+x^6}\,dx$$



We compute that

$$\left|\int_{S_R} \frac{dz}{1+z^6}\right| \leq \pi R \left(\frac{1}{R^6-1}\right) \text{ for large enough } R.$$

$$\to 0 \text{ as } R \to \infty.$$

$$\int_{-R}^{R} \frac{1}{1+x^6} \to \int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx \text{ as } R \to \infty.$$

$$\int_{C_R} \frac{1}{1+z^6} = 2\pi i \sum \left(\text{residue of } \frac{1}{1+z^6} \text{ inside } C_R\right)$$

If $1+z^6=0$

$$z = e^{i(\pi+2k\pi)\frac{1}{6}} \quad k=0,1,2\ldots,5$$

where $e^{i\pi/6}$, $e^{i\pi/2}$, $e^{5\pi i/6}$ are inside C_R, they are simple pole. To observe this, we let the roots of $1+z^6$ be $z_k = e^{i\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}$

$$\frac{1}{1+z^6} = \prod_{k=1}^{6}\left(\frac{1}{z-z_k}\right)$$

Around the point z_j, $\frac{1}{1+z^6} = \frac{1}{z-z_j}\left(\prod_{\substack{k=1\\k\neq j}}^{6} \frac{1}{z-z_k}\right)$

$\prod_{\substack{k=1 \\ k\neq j}}^{6} \frac{1}{z-z_k}$ is analytic at $z=z_j$. Thus each pole is simple. To compute the residue of $\frac{1}{1+z^6}$ at these poles, we found that

$$\operatorname*{Res}_{z=z_k} \frac{1}{1+z^6} = \lim_{z\to z_k}\left(\frac{z-z_k}{1+z^6}\right)$$

$$= \frac{1}{6z_k^5}.$$

Therefore, $\int_{C_R} \frac{1}{1+z^6} = \frac{2\pi i}{6}\left(e^{-5i\pi/6} + e^{-5i\pi/2} + e^{-25\pi i/2}\right)$

$$= \frac{2\pi}{3}$$

Taking $R\to\infty$, $\int_{-\infty}^{\infty} \frac{1}{1+x^6} = \frac{2\pi}{3}$.

(2) $\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{\pi e^{-1}}{2}$

Ans: observe that $2\int_0^\infty \frac{\cos x}{x^2+1} = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$ since the integrand is even. We use the closed path in (1). We consider

$$\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^2} + \int_{S_R} \frac{e^{iz}}{1+z^2}$$

$$\left| \int_{SR} \frac{e^{iz}}{1+z^2} \right| \leq \frac{\pi}{R^2-1} \quad \text{by Jordan Lemma.}$$

$$\to 0 \quad \text{as } R \to \infty.$$

$$\int_{-R}^{R} \frac{e^{ix}}{1+x^2} dx \to \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \quad \text{as } R \to \infty$$

$$\int_{CR} \frac{e^{iz}}{1+z^2} = \int_{CR} \left(\frac{e^{iz}}{z+i} \right) \left(\frac{1}{z-i} \right)$$

$$= 2\pi i \left(\frac{e^{-1}}{2i} \right) \quad \text{by Cauchy integral formula.}$$

$$= \pi e^{-1}$$

Therefore $\displaystyle\int_{CR} \frac{e^{iz}}{1+z^2} = \int_{-R}^{R} \frac{e^{ix}}{1+x^2} + \int_{SR} \frac{e^{iz}}{1+z^2}$,

taking $R \to \infty$,

$$\pi e^{-1} = \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{1+x^2}$$

Comparing the Real part,

$$\int_0^{\infty} \frac{\cos x}{x^2+1} = \frac{\pi e^{-1}}{2}.$$

③ $\int_0^{\pi} \frac{1}{3+2\cos\theta} d\theta = \frac{\pi}{\sqrt{5}}$

Ans : Due to the property of $\cos\theta$, we see that

$$\int_0^{\pi} \frac{1}{3+2\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{3+2\cos\theta} d\theta .$$

(It is because $\int_0^{\pi} \frac{1}{3+2\cos\theta} = \int_{\pi}^{2\pi} \frac{1}{3+2\cos\theta}$).

If we consider $z = e^{i\theta}$ for $\theta \in (0, 2\pi]$,

$$\int_0^{2\pi} \frac{1}{3+2\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{3+\left(z+\frac{1}{z}\right)} \frac{1}{ie^{i\theta}} de^{i\theta}$$

$$= \int_C \frac{dz}{(3z+z^2+1)i}, \text{ C is unit circle.}$$

If $3z+z^2+1 = 0$, then $z = \frac{-3 \pm \sqrt{5}}{2}$. The only pole in C is $z = \frac{-3+\sqrt{5}}{2}$. Therefore,

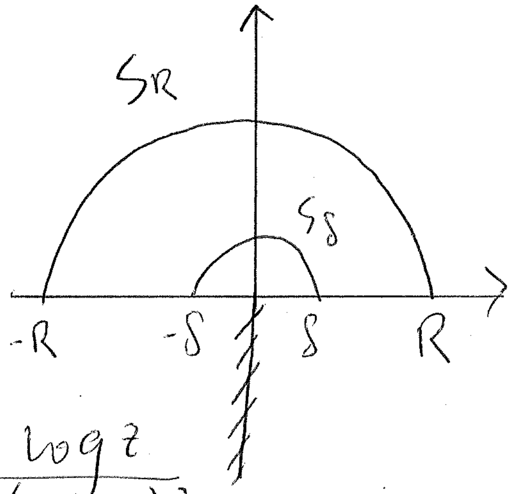
$$\int_C \frac{dz}{3z+z^2+1} = 2\pi i \frac{1}{\sqrt{5}} \left(\frac{1}{i}\right)$$

$$= \frac{2\pi}{\sqrt{5}}$$

Thus $\int_0^{\pi} \frac{1}{3+2\cos\theta} d\theta = \frac{\pi}{\sqrt{5}}$.

④ $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2} = -\frac{\pi}{4}.$

Ans : We consider the closed contour $C_{R,\delta}$ and take log with



$\cancel{\displaystyle\oint_{C_R/\delta} \frac{\log z}{(1+z^2)^2}}$ branch $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$$\int_{C_{R,\delta}} \frac{\log z}{(1+z^2)^2} = \int_{S_R} \frac{\log z}{(1+z^2)^2} + \int_{S_\delta} \frac{\log z}{(1+z^2)^2}$$

$$+ \int_\delta^R \frac{\log x}{(1+x^2)^2} + \int_{-R}^{-\delta} \frac{\log x}{(1+x^2)^2}$$

$$\int_{-R}^{-\delta} \frac{\log x}{(1+x^2)^2} = -\int_R^\delta \frac{\log(-y)}{(1+y^2)^2}\,dy \quad (x=-y)$$

$$= \int_\delta^R \frac{\log y + i\pi}{(1+y^2)^2}\,dy$$

Thus, $\displaystyle\int_\delta^R + \int_R^{-\delta} \frac{\log x}{(1+x^2)^2} = 2\int_\delta^R \frac{\log x}{(1+x^2)^2} +$

$$i\pi \int_\delta^R \left(\frac{1}{1+x^2}\right)^2 dx$$

$$\left| \int_{S_R} \frac{\log z}{(1+z^2)^2} \right| \leq \pi R \frac{\log R + \pi}{(R^2-1)^2} \to 0 \text{ as } R \to \infty$$

$$\left| \int_{S_\delta} \frac{\log z}{(1+z^2)^2} \right| \leq \pi \delta \left(\frac{|\log \delta| + \pi}{(1-\delta^2)^2} \right) \to 0 \text{ as } \delta \to 0.$$

$$\int_{C_{R,\delta}} \frac{\log z}{(1+z^2)^2} = \cancel{2\pi \mathrm{Res}}\ 2\pi i \operatorname*{Res}_{z=i} \frac{\log z}{(1+z^2)^2}$$

since $\log z$ is analytic at $z=i$ and ~~Res~~ $\cancel{\frac{\log z}{(1+z^2)^2}}$ ~~≠~~ the pole order of

$\frac{\log z}{(1+z^2)^2}$ is 2. Then

$$\operatorname*{Res}_{z=i} \frac{\log z}{(1+z^2)^2} = \frac{d}{dz} (z-i)^2 \left(\frac{\log z}{(z^2+1)^2} \right) \Bigg|_{z=i}$$

$$= \frac{d}{dz} \left(\frac{\log z}{(z+i)^2} \right) \Bigg|_{z=i}$$

$$= \frac{\pi + 2i}{8}$$

Thus $2\pi i \left(\frac{\pi+2i}{8} \right) = 2\int_0^\infty \frac{\log x}{(1+x^2)^2} + i\pi \int_0^\infty \left(\frac{1}{1+x^2} \right)^2$ as $R \to \infty$, $\delta \to 0$.

$\Rightarrow \int_0^\infty \frac{\log x}{(1+x^2)^2} = -\frac{\pi}{4}$ by comparing the real part.