① $e^{z-1} = \sum_{n\geq 0}^{\infty} \frac{1}{n!}(z-1)^n$

$e^{z} = \sum_{n\geq 0}^{\infty} \frac{e}{n!}(z-1)^n$

② The possible singularity is $z=0$

Method A: $\lim_{z\to 0} z \cdot \left(\frac{\cos z - 1}{z}\right) = 0$

Method B: $\cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$

Laurent series of f =

$\frac{\cos z - 1}{z} = -\frac{z}{2!} + \frac{z^3}{4!} + \ldots$

③ $z e^{\frac{1}{z}} = z \cdot \sum_{n\geq 0}^{\infty} \frac{1}{n!\, z^n}$

$= z + \sum_{n\geq 0}^{\infty} \frac{1}{(n+1)!\, z^n}$

$z=0$ is essential singularity.

④ Since $\cos^3 z$ is analytic,

$\cos^3 z = a_0 + a_1 z + a_2 z^2 + \ldots$

for some $a_0, a_1, a_2 \ldots \in \mathbb{C}$.

Thus $\frac{\cos^3 z}{z} = \frac{a_0}{z} + a_1 + a_2 z + \ldots$

The order of pole is 1.

(5) let the order of pole at $z = a$ be m

we write $(z-a)^m f = p(z) + (z-a)^m h$

with h and p and analytic, $\deg(p) < m$,

$$e^f = e^{\frac{p}{(z-a)^m}} e^h$$

since $\deg(p) < m$, $e^{\frac{p}{(z-a)^m}}$ admits an essential singularity at a.