① let C_R be circle with radius R and centred at 0,

We consider $\int_{C_R} \frac{f}{z^3} = \int_{C_R} \frac{a_0}{z^3} + \int_{C_R} \frac{a_1}{z^2} + \int_{C_R} \frac{a_2}{z} + \cdots$

Since all the terms have antiderivative except $\frac{a_2}{z}$, thus,

$$\int_{C_R} \frac{a_2}{z} = \int_{C_R} \frac{f}{z^3}$$

$$\left| a_2 (2\pi i) \right| \leq \int_{C_R} \frac{|f|}{|z^3|}$$

$$\leq \frac{A}{R^2} \cdot 2\pi R \to 0 \text{ as } R \to \infty,$$

We consider $\int_{C_R} \frac{f}{z^4} = \cdots$

thus $a_n = 0 \ \forall n \geq 2$

Therefore $f = a_0 + a_1 z$, since $f(0) = 0$, then

$f = a_1 z$.

② $e^f = e^u \cdot e^{iv}$

$|e^f| = |e^u| \le e^M$,

By Liouville's theorem, e^f is constant.

$\Rightarrow f$ is constant.

③ Since f is analytic and $f \neq 0$ in $\overline{U}$,

$\frac{1}{f}$ is also analytic, then by Max. P.

$\frac{1}{|f|}$ can not attain its max. in U.

$\Rightarrow |f|$ can not attain its min in U.

④ We consider $e^f = e^u \cdot e^{iv}$, by Max. P.

$|e^f| = e^u$ can not attain its max in R since

e^f is analytic. Thus u can not attain its max.

in R also, since e^x is increasing.

Thus u attains it max and min on boundary of

R due to continuity of f and ex. ③

Since $u = e^x \cos y$, on $x = 0$, $\max u = 1$ (at $(0,0)$)

on $y = 0$, $\max u = e$ (at $(1,0)$)

on $x=1$, $\max u = e$ (at $(1,0)$)

on $y=\pi$, $\max u = -1$ (at $(0,\pi)$).

$\max u = e$ at $z=1$

Similarly, $\min u = -e$ at $z = 1+\pi i$