Optimization Theory

Tutorial 8

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Table of Contents

Table of Contents

Concepts

Direction of recession of C: we say that a vector d is a direction of recession of C if $x + \alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$.

Recession of cone of C: the set of all directions of recession is said to be recession of cone of C. It is a cone containing the origin. It is denoted by R_C.

Lineality space: the set of direction of recession d whose opposite, $-d$, are also directions of recession:

$$L_C = R_C \cap (-R_C).$$

It is denoted by L_C. Thus $d \in L_C$ if and only if the entire line $\{x + \alpha d | a \in \Re\}$ is contained in C for every $x \in C$.

Epigraph: the epigraph of a function $f : X \to [-\infty, \infty]$, where $X \subset \Re^n$, is defined to be the subset of $\Re^{n+1}$ given by

$$epi(f) = \{(x, w) | x \in X, w \in \Re, f(x) \leq w\}.$$

Theorem

Saddle Point: *A pair of vectors $x^* \in X$ and $z^* \in Z$ is called a saddle point of ϕ if*

$$\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*), \forall x \in X, \forall z \in Z.$$

minimax equality:

$$sup_{z\in Z} inf_{x\in X} \phi(x, z) = inf_{x\in X} sup_{z\in Z} \phi(x, z).$$

Saddle Point and Minimax Theory

Theorem

A pair (x^*, z^*) is a saddle point of ϕ if and only if the minimax equality holds, and x^* is an optimal solution of the problem:

$$\text{minimize} \sup_{z \in Z} \phi(x, z), \text{ subject to } x \in X,$$

while z^* is an optimal solution of the problem

$$\text{maximize} \inf_{x \in X} \phi(x, z), \text{ subject to } z \in Z$$

Recession Cone Theorem

Let C be a nonempty closed convex set.

(a) The recession cone R_C is closed and convex.

(b) A vector d belongs to R_C if and only if there exists a vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

Properties of Recession Cones

Let C be a nonempty closed convex set.

(a) R_C contains a nonzero direction if and only if C is unbounded.

(b) $R_C = R_{ri(C)}$.

(c) For any collection of closed convex sets $C_i, i \in I$, where I is an arbitrary index set and $\cap_{i \in I} C_i \neq \emptyset$, we have

$$R_{\cap_{i \in I} C_i} = \cap_{i \in I} R_{C_i}.$$

(d) Let W be a compact and convex subset of $\Re^m$, and let A be an $m \times n$ matrix. The recession cone of the set

$$V = \{x \in C | Ax \in W\}$$

(assuming this set is nonempty) is $R_C \cap N(A)$, where $N(A)$ is the nullspace of A.

Properties of Lineality Space

Let C be a nonempty closed convex set of $\Re^n$.

(a) L_C is a subspace of $\Re^n$.

(b) $L_C = L_{ri(C)}$.

(c) For any collection of closed convex sets $C_i, i \in I$, where I is an arbitrary index set and $\cap_{i \in I} C_i \neq \emptyset$, we have

$$L_{\cap_{i \in I} C_i} = \cap_{i \in I} L_{C_i}.$$

(d) Let W be a compact and convex subset of $\Re^m$, and let A be an $m \times n$ matrix. The lineality space of the set

$$V = \{x \in C | Ax \in W\}$$

(assuming this set is nonempty) is $L_C \cap N(A)$, where $N(A)$ is the nullspace of A.

Solution

Using

$$L_C = R_C \cap (-R_C).$$

and Properties of R_C.

Decomposition of a convex set

Let C be a nonempty subest of $\Re^n$. Then, for every subspace S that is contained in the lineality space L_C, we have

$$C = S + (C \cap S^\perp).$$

Decomposition of a convex set

Let C be a nonempty subest of $\Re^n$. Then, for every subspace S that is contained in the lineality space L_C, we have

$$C = S + (C \cap S^{\perp}).$$

Proof: We can decompose $\Re^n$ as $S + S^{\perp}$, so for $x \in C$, let $x = d + z$ for some $d \in S$ and $z \in S^{\perp}$. Because $-d \in S \subset L_C$, the vector $-d$ is a direction of recession of C, so the vector $x - d$, which is equal to z, belongs to C, implying that $z \in C \cap S^{\perp}$. Thus, we have $x = d + z$ with $d \in S$ and $z \in C \cap S^{\perp}$ showing that $C \subset S + (C \cap S^{\perp})$.

Conversely, if $x \in S + (C \cap S^{\perp})$, then $x = d + z$ with $d \in S$ and $z \in C \cap S^{\perp}$. Thus, we have $z \in C$. Furthermore, because $S \subset L_C$, the vector d is a direction of recession of C, implying that $d + z \in C$. Hence $x \in C$, showing that $S + (C \cap S^{\perp}) \subset C$. **Q.E.D.**

Let $f : \Re^n \to (-\infty, \infty]$ be a closed proper convex function and consider the level sets

$$V_\gamma = \{x | f(x) \leq \gamma\}, \quad \gamma \in \Re.$$

Then:

(a) All the nonempty level sets V_γ have the same recession cone, denoted R_f, and given by

$$R_f = \{d | (d, 0) \in R_{epi(f)}\},$$

where $R_{epi(f)}$ is the recession cone of the epigraph of f.

(b) If one nonempty level set V_γ is compact, then all of these level sets are compact.

Solution

(a) Fix a γ such that V_γ is nonempty, consider

$$S = \{(x, \gamma) | f(x) \leq \gamma\},$$

$$S = epi(f) \cap \{(x, r) | x \in \Re^n\}.$$

$$R_S = R_{epi(f)} \cap \{(d, 0) | d \in \Re^n\} = \{(d, 0) | (d, 0) \in R_{epi(f)}\},$$

independent of γ.

(b)

$$V_\gamma \text{ compact} \Leftrightarrow R_{V_\gamma} \text{does NOT contain a nonzero direction}$$

$$\Rightarrow R_{V_{\gamma_1}} \text{does NOT contain a nonzero direction}$$

$$\Rightarrow V_{\gamma_1} \text{ compact.}$$