Tutorial 6 10\10\2014

This tutorial will focus on shape operator.

Recall : M = a regular surface in $\mathbb{R}^3$

$p \in M$.

We def the shape operator for M at p by

$$S_p(v) = -\nabla_v U \quad \forall\, v \in T_pM$$

where U = the unit normal vector field on M.

We have shown S_p is a linear transformation from T_pM into itself.

For $\mathbb{R}^2$, $\forall p \in \mathbb{R}^2$, $S_p = 0$.

For S^2,

$$X(u,v) = (\cos u \cos v, \sin u \cos v, \sin v)$$

$$\Rightarrow \quad U = \frac{X_u \times X_v}{|X_u \times X_v|} = (\cos u \cos v, \sin u \cos v, \sin v)$$

$$S_p(X_u) = -\nabla_{X_u} U = -\sum_i X_u(U^i)\, e_i \qquad \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{e_i = (0, \overset{i\text{th}}{1}, 0)}$$

$$= -(-\sin u \cos v, \cos u \cos v, 0)$$

$$= -X_u$$

Similarly, $S_p(X_v) = -X_v$.

So $\forall\, v \in T_pM$ $v = aX_u + bX_v$ $\rightarrow$ $S_p(v) = -v$

So $(S_p = -id_{T_pM})$

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Next we show S_p is a symmetric linear operator.

Actually, let $X(u,v)$ be a coordinate patch around p.

Want to show

$$\begin{cases} \langle S_p(X_u), X_u\rangle = \langle X_{uu}, U\rangle & ① \\ \langle S_p(X_u), X_v\rangle = \langle X_{uv}, U\rangle & ② \\ \langle S_p(X_v), X_v\rangle = \langle X_{vv}, U\rangle & ③ \end{cases}$$

Here we only show ② (So S_p is symmetric since $X_{uv} = X_{vu}$ if X is C^∞.)

$$\langle X_u, U\rangle = 0$$

$$X_v\langle X_u, U\rangle = 0$$

$$\text{LHS} = \langle X_{uv}, U\rangle + \langle X_u, \nabla_v U\rangle$$

$$\begin{aligned} \text{So } \langle S_p(X_v), X_u\rangle &= \langle X_{uv}, U\rangle \\ &= \langle X_{vu}, U\rangle \\ &= \langle S_p(X_u), X_v\rangle. \end{aligned}$$

Matrix representation of $S_p : T_pM \to T_pM$ w.r.t $[X_u, X_v]$.

Let's use modern notation $[X_u, X_v] \Leftrightarrow [X_1, X_2]$.

$$S_p(X_j) = \sum_{i=1}^{2} a_j^i X_i$$

Want to compute $A = [a_j^i]$.

By the above thm, $\langle S_p(X_i), X_j \rangle = \langle X_{ij}, U \rangle$

$$\Rightarrow \quad a_i^k \langle X_k, X_j \rangle = \langle X_{ij}, U \rangle.$$

$g_{ij} \triangleq \langle X_i, X_j \rangle$, $h_{ij} \triangleq \langle X_{ij}, U \rangle$

$G = (g_{ij})$

$$a_i^k g_{kj} = h_{ij}$$

$$a_i^k g_{kj} g^{js} = g^{js} h_{ij} \quad \text{where } g^{ij} = G^{-1}\text{'s component}$$

$$a_i^k \delta_{ks} = g^{js} h_{ij}$$

$$a_i^s = g^{js} h_{ij}$$

$$\Rightarrow \quad A = G^{-1} \cdot H.$$

Classical notation:

$$E = \langle X_u, X_u \rangle = g_{11}$$

$$F = \langle X_u, X_v \rangle = g_{12} = g_{21}$$

$$G = \langle X_v, X_v \rangle = g_{22}$$

$$l = \langle X_{uu}, U \rangle = h_{11}$$

$$m = \langle X_{uv}, U \rangle = h_{12} = h_{21}$$

$$n = \langle X_{vv}, U \rangle = h_{22}$$

$$G = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad H = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$G^{-1} = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = (g^{ij})$$

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} = G^{-1}H = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$= \frac{1}{EG-F^2} \begin{pmatrix} Gl - Fm & Gm - Fn \\ -Fl + Em & -Fm + En \end{pmatrix}$$

Note that G^{-1} exists since M is regular.

$$EG - F^2 \neq 0.$$

$$\Leftrightarrow \quad |X_u|^2 |X_v|^2 - \langle X_u, X_v \rangle^2 \neq 0$$

$$\Leftrightarrow \quad |X_u \times X_v|^2 \neq 0.$$

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Thm: If $S_p \equiv 0 \ \forall p \in M$, then M is contained in a plane

Pf: Fix $p \in M$. $U(p)$ = unit normal of p.

For any $q \in M$, $\exists\, \alpha(t)$ curve $0 \le t \le 1$ on M s.t.

$$\alpha(0) = p, \quad \alpha(1) = q.$$

Consider

$$f(t) = \langle \alpha(t) - q,\ U(\alpha(t)) \rangle$$

$$f'(t) = \cancelto{0}{\langle \underbrace{\alpha'(t)}_{\text{tangent}},\ \underbrace{U(\alpha(t))}_{\text{normal}} \rangle} + \cancelto{0}{\langle \alpha(t) - q,\ \nabla_{\alpha'(t)} U(\alpha(t)) \rangle}$$

$$= 0.$$

So $f \equiv$ Constant.

$$\langle p - q, U(p) \rangle = f(0) = f(1) = \langle q - q, U(q) \rangle = 0.$$

We have shown for $q \in M$

$$\langle p-q, U(p) \rangle = 0$$

So $M \subset$ a plane. #

So S_p measures the different of a surface, in shape, from being a plane.

Let's def the differential (Tangent map).

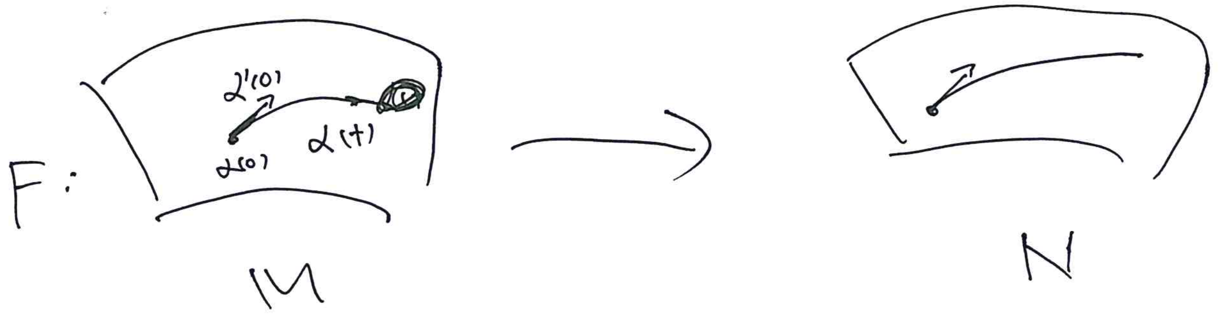
M, N = regular surface

$F: M \to N$ = C^∞ map.

The differential of F (DF, F_*) at $p \in M$.

$$DF_p = (F_*)_p : T_pM \to T_{F(p)}N$$

$$DF_p(\alpha'(0)) = \frac{d}{dt}\Big|_{t=0} F(\alpha(t)) \quad \forall \alpha'(0) \in T_pM.$$



The shape operator can be consider as a differential.

$$U : M \to S^2$$

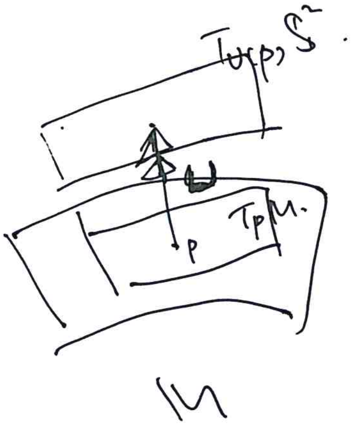
For any $v \in T_pM$. α is a curve with $\alpha(0) = p$, $\alpha'(0) = v$.

$$(U_*)_p(v) = \frac{d}{dt}\Big|_{t=0} \big(U(\alpha(t))\big)$$

$\cap$

$$T_{U(p)}S^2 \qquad = \nabla_{\alpha'(0)} U$$

$$= -S_p(v) \quad \in T_pM.$$



$$T_pM \parallel T_{U(p)}S^2$$