Tutorial 5 03/10/2014

(1) Let $V(t) = (x(t), y(t), z(t))$ be a 3-time differentiable vector-valued function def. on an open interval I (containing $t=0$) s.t $|V|, |v \times v'|$, $\langle v \times v', v'' \rangle$ are non-vanishing functions. Find the torsion of the curve $\alpha : I \to \mathbb{R}^3$ def. by

$$\alpha(t) = \left(\int_0^t \frac{yz' - y'z}{|v|^2} dt, \int_0^t \frac{zx' - z'x}{|v|^2} dt, \int_0^t \frac{xy' - x'y}{|v|^2} dt \right).$$

{ If α is a regular curve s.t $k > 0$, then

$$\tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}. \quad \left(\tau \triangleq -\langle B', N \rangle \text{ (parametrized by arc-length)} \right).$$

$$\alpha' = \left(\frac{yz' - y'z}{|v|^2}, \frac{zx' - z'x}{|v|^2}, \frac{xy' - x'y}{|v|^2} \right)$$

$$= \frac{v \times v'}{|v|^2} \neq 0 \quad \underline{\underline{\text{regular}}}$$

One may def $n \triangleq \frac{v}{|v|}$, $\alpha' = n \times n'$ $\left(n \times n' = \frac{v}{|v|} \times \left(\frac{v'}{|v|} + \left(\frac{1}{|v|}\right)' v \right) \right)$

$\alpha'' = n \times n''$. $\alpha' \times \alpha'' = (n \times n') \times (n \times n'')$

$$\boxed{\begin{aligned} &a \times (b \times c) \\ &= b(a \cdot c) - c(a \cdot b) \end{aligned}}$$

$$= \langle n \times n', n'' \rangle n - \langle n \times n', n \rangle n'' \quad (\langle n \times n', n \rangle = 0)$$

$\Rightarrow$ $|\alpha' \times \alpha''|^2 = \langle n \times n', n'' \rangle^2$.

$|\langle n \times n', n \rangle|$ = Vol of the parallelepiped spanned by $\{n, n', n\}$.

$\alpha''' = n' \times n'' + n \times n'''$

$$\langle \alpha' \times \alpha'', \alpha''' \rangle$$

$$= \left\langle \langle n \times n', n'' \rangle n, \; n' \times n'' + n \times n''' \right\rangle$$

$$= \langle n \times n', n'' \rangle^2 .$$

So $$\tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2} = 1 .$$

Finally, we should check $\kappa > 0$.

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

$$\langle n \times n', n'' \rangle = \left\langle \frac{v \times v'}{|v|^2}, \left(\frac{v'}{|v|} + \left(\frac{1}{|v|}\right)' v \right)' \right\rangle$$

$$= \frac{1}{|v|^2} \left\langle v \times v', \frac{v''}{|v|} + 2\left(\frac{1}{|v|}\right)' v' + \left(\frac{1}{|v|}\right)'' v \right\rangle$$

$$= \frac{1}{|v|^3} \langle v \times v', v'' \rangle \neq 0 .$$

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(2) Let M be a surface given by $X(u,v) = (au, bv, (u+v)^2 + (u-v)^2)$ where $u, v \in \mathbb{R}$ and a, b are non-zero real constants.

Find the equation of the tangent plane $T_p(M)$ to M at $p = (au, 0, 2u^2)$ $\forall u \in \mathbb{R}$, in the form of $cx + dy + ez = f$.

$$X_u \times X_v = (-4bu, -4av, ab)$$

At $p = (au, 0, 2u^2)$ (i.e. $v=0$)

$$X_u \times X_v = (-4bu, 0, ab)$$

$$\langle (x,y,z) - (au, 0, 2u^2), (-4bu, 0, ab) \rangle = 0.$$

$$-4bux + abz = -4abu^2 + 2u^2ab$$

$$-4ux + az = -4au^2 + 2u^2a = -2u^2a$$

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(3) Let $\alpha(t)$ be a unit speed curve with torsion $\tau_\alpha > 0$. Let $B_\alpha(t)$ be the binomal of α at $\alpha(t)$. Def. a curve $\beta(s) = \int_0^s B_\alpha(t)\,dt$.
Show that $\kappa_\beta = \tau_\alpha$, $\tau_\beta = \kappa_\alpha$, $T_\beta = B_\alpha$, $N_\beta = -N_\alpha$ and $B_\beta = T_\alpha$
where $[T_{\alpha(\beta)}, N_{\alpha(\beta)}, B_{\alpha(\beta)}]$, $\kappa_{\alpha(\beta)}, \tau_{\alpha(\beta)}$ are frenet frame, curvature and torsion of $\alpha(\beta)$.

Note that $\beta(s)$ is also a unit speed curve.

$$T_\beta = \beta' = B_\alpha \qquad B_\beta = T_\beta \times N_\beta = B_\alpha \times (-N_\alpha) = T_\alpha$$

$$T_\beta' = B_\alpha' = -\tau_\alpha N_\alpha \qquad \tau_\beta = -\langle B_\beta', N_\beta \rangle$$

$$\kappa_\beta = |T_\beta'| = \tau_\alpha \qquad = -\langle T_\alpha', -N_\alpha \rangle$$

$$N_\beta = \frac{T_\beta'}{\kappa_\beta} = \frac{-\tau_\alpha}{\tau_\alpha} N_\alpha = -N_\alpha \qquad = \langle T_\alpha', N_\alpha \rangle$$

$$= \kappa_\alpha.$$

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④ Q4 in Assignment 2 :

Let α be a smooth closed plane curve in $\mathbb{R}^2$ which is contained in a disk of radius $r > 0$. Prove that $\exists\, p \in \alpha$ s.t. $k(p) \geq \frac{1}{r}$.

W.L.O.G, we assume $\alpha(t)$ is parametrized by arc-length and the disk has centre at origin. We consider the farthest pt on α (w.r.t the origin)
↓ say $\alpha(t_0)$.

$f(t) \triangleq \|\alpha(t)\|^2$ (smooth function)

So we have $\begin{cases} f'(t_0) = 0 \\ f''(t_0) \leq 0 \end{cases}$ (We can parametrise s.t $t_0 \in (0, l)$ where l = length of α)

$$f'' = 2\left(\langle \alpha'', \alpha \rangle + \langle \alpha', \alpha' \rangle \right)$$

$$\Rightarrow \quad \langle \alpha''(t_0), \alpha(t_0) \rangle \leq -1.$$

$$\Rightarrow \quad k(t_0) \cdot d \cos\theta \leq -1 \qquad d = \mathrm{dist}(0, \alpha(t_0))$$

$$\Rightarrow \quad \cos\theta < 0$$

$$k(t_0) \geq \frac{-1}{d\cos\theta} \geq \frac{-1}{r\cos\theta} \geq \frac{1}{r}.$$

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⑤ Let $\beta(s)$, $s \in I$, be a unit speed curve with nonvanishing curvature $k > 0$. Let $X(s, v) = \beta(s) + r\left(N(s)\cos v + B(s)\sin v\right)$, $s \in I$, $v \in (0, 2\pi)$ where r is a non-zero constant; $N(s)$ and $B(s)$ are principal normal and binormal of $\beta(s)$ respectively.

(i) When $X_s \times X_v \neq 0$?

(ii) Show that when $X_s \times X_v \neq 0$, say at $X(s, v)$, the surface is normal to the vector

$$\underbrace{N(s)\cos v + B(s)\sin v}$$

(iii) When is this vector equals to $X_s \times X_v / |X_s \times X_v|$?

Ex 5:

(i) By direct computation and Frenet's formula, we have

$$X_s = \beta' + r(N'\cos v + B'\sin v)$$

$$= T + r[(-\kappa T + \tau B)\cos v - \tau N \sin v]$$

$$= (1 - r\kappa\cos v)T - r\tau\sin v N + r\tau\cos v B$$

$$X_v = r(-N\sin v + B\cos v)$$

$$= -r\sin v N + r\cos v B$$

$$\Rightarrow \quad X_s \times X_v = -(1-r\kappa\cos v)r\sin v B - (1-r\kappa\cos v)r\cos v N \quad (*)$$

$$- r^2\tau\cancel{\sin v\cos v}\,T + r^2\tau\cancel{\sin v\cos v}\,T$$

Then $X_s \times X_v \neq 0$ iff $\begin{cases} 1 - r\kappa\cos v \neq 0 \\ \sin v \neq 0 \text{ or } \cos v \neq 0. \end{cases}$

i.e

$$\begin{cases} \cos v \neq \frac{1}{r\kappa(s)} \\ \sin v \neq 0 \text{ or } \cos v \neq 0. \end{cases} \Rightarrow \quad \cos v \neq \frac{1}{r\kappa(s)}$$

(ii) When X is regular, by $(*)$ above, we have

$$(X_s \times X_v) \times (N\cos v + B\sin v)$$

$$= (1-r\kappa\cos v)r\sin v\cos v\,T - (1-r\kappa\cos v)r\cos v\sin v\,T = 0. \quad \#$$

(iii) One may further compute:

$$\frac{X_s \times X_v}{|X_s \times X_v|} = \frac{-(1-rk)r\cos v}{|1-rk\cos v|\cdot|r|} N - \frac{(1-rk\cos v)r\sin v}{|1-rk\cos v|\cdot|r|} B$$

Then $\dfrac{X_s \times X_v}{|X_s \times V_v|} = N\cos v + B\sin v$

iff

$$\left(-1 - \frac{(1-rk\cos v)r}{|1-rk\cos v|\cdot|r|}\right)\cos v = 0$$

and

$$\left(-1 - \frac{(1-rk\cos v)r}{|1-rk\cos v|\cdot|r|}\right)\sin v = 0$$

iff

$$-1 - \frac{(1-rk\cos v)r}{|1-rk\cos v|\cdot|r|} = 0$$

iff $-(1-rk\cos v)r > 0$.

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