Tutorial 4 26/09/2014

(1) An exercise in last tutorial :

$X(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$ (General form of surfaces of revolution) (corresponding to x-axis)

Show that ~~X is regular~~ $X_u \times X_v \neq 0 \ \forall (u,v)$ $\Leftrightarrow$ $\alpha(u) = (g(u), h(u), 0)$ is regular and has no intersection with x-axis.

Pf: ($\Rightarrow$) We compute $\begin{cases} X_u = (g', h'\cos v, h'\sin v) \\ X_v = (0, -h\sin v, h\cos v) \end{cases}$

$\Rightarrow X_u \times X_v = (h'h, -g'h\cos v, -g'h\sin v)$

And compute $\alpha' = (g', h', 0)$

If $g' = h' = 0$, then $X_u \times X_v = 0$ $\rightarrow\leftarrow$

If $h = 0$, then $X_u \times X_v = 0$ $\rightarrow\leftarrow$.

So α is regular and has no intersection with x-axis.

($\Leftarrow$) Assume $\downarrow$, we have

$h \neq 0$ and g', h' can't equ. to 0 at the same time.

If $h' = 0$, $X_u \times X_v = (0, -g'h\cos v, -g'h\sin v) \neq 0$

If $g' = 0$, $X_u \times X_v = (h'h, 0, 0) \neq 0$

If $h' \neq 0$ and $g' \neq 0$, $X_u \times X_v \neq 0$.

So we have $X_u \times X_v \neq 0$.

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② Ex: Let $M : X(u,v) = \beta(u) + v\delta(u)$ be a ruled surface with $|\beta'| \equiv 1$ and $|\delta| \equiv 1$. Show that if $\delta' \neq 0$ $\forall u$, M may be reparametrized by $Y(z,w) = r(z) + w\delta(z)$ s.t. $\langle r', \delta' \rangle = 0$.
$(= X(u,v))$

One may consider $r(u) = \beta(u) + s(u)\delta(u)$ where $s(u)$ is a C^∞ function on

Then $r' = \beta' + s'\delta + s\delta'$. Using $\langle r', \delta' \rangle = 0$,

$$\langle \beta', \delta' \rangle + s' \cancel{\langle \delta, \delta' \rangle} + s|\delta'|^2 = 0$$

Since $|\delta| = 1$

$$s(u) = \frac{-\langle \beta', \delta' \rangle}{|\delta'|^2}$$ Note that $\delta' \neq 0$ $\forall u$.

Let $\begin{cases} z = u \\ w = v - s(u) \end{cases}$

$$Y(z,w) \overset{\triangle}{=} r(z) + w\delta(z)$$
$$= \beta(u) + s(u)\delta(u) + (v - s(u))\delta(u)$$
$$= X(u,v).$$

At last, we should check

$$\det\begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ -s' & 1 \end{pmatrix} = 1 \neq 0$$

(It's actually a coordinate change!)

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(3) We may def the tangent plane T_pM of M at p in this way (Global one) :

$$T_pM \triangleq \left\{ \alpha'(0) \;\middle|\; \begin{array}{l} \alpha : (-\varepsilon, \varepsilon) \to M \\ \alpha(0) = p \end{array}, \text{ for some } \varepsilon > 0 \right\}$$

Elements in T_pM are called tangent vectors to M at p.

One can show the following lemma (Lemma 2.2.1 of Oprea)

Let
- M = regular surface
- $p \in M$
- $X(u,v)$ = regular patch on M s.t. $p = X(u_0, v_0)$

Then T_pM is a vector space with basis $\{X_u, X_v\}_{(u_0,v_0)}$

Let's compute some examples.

M = graph of "$z = f(x,y)$"

Monge patch : $X(u,v) = (u, v, f(u,v))$

$$\Rightarrow \begin{cases} X_u = (1, 0, f_u) \\ X_v = (0, 1, f_v) \end{cases}$$

$$\Rightarrow \quad X_u \times X_v = (-f_u, -f_v, 1) \quad = \text{normal} \quad (\text{may be not unit})$$

$$p = (u_0, v_0, f(u_0, v_0))$$

$$\left\langle (x, y, z) - p \,,\, X_u \times X_v \Big|_{(u_0, v_0)} \right\rangle = 0.$$

i.e the equ. of T_pM

$$-f_u(u_0, v_0)(x - u_0) - f_v(u_0, v_0)(y - v_0) + (z - f(u_0, v_0)) = 0.$$

Explicit eq: $\quad f(x, y) = x^2 + y^2 \quad$ (Paraboloid)

$$p = (1, 0, 1)$$

$$f_u = 2u$$

$$f_v = 2v$$

$$-2(x - 1) + z - 1 = 0$$

i.e

$$2x - z = 1.$$

④ Recall: The Gauss map is a unit normal vector field on M

i.e $U : M \to S^2$. In the case of a coordinate patch $X(u, v)$

$$U = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

If we write $\mathbb{U} = u^1 e_1 + u^2 e_2 + u^3 e_3$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then u^1, u^2, u^3 are functions on M.

For any $v \in T_pM$ $\left(\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}\right)$, we denote

$$\nabla_v \mathbb{U} \triangleq \sum_{i=1}^{3} v[u^i] e_i$$

directional derivative

$v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$

$v(f) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$.

Def: The shape operator (or Weingarten Map) of a surface M at p is def. by

$$S_p(v) = -\nabla_v \mathbb{U}, \quad \forall v \in T_pM$$

where $\mathbb{U}$ is the Gauss map of M.

Lemma: S_p is a linear transformation from T_pM into itself

Pf: (1) Want $\nabla_v \mathbb{U} \in T_pM$.

$|\mathbb{U}| = 1$

$$0 = v[\langle \mathbb{U}, \mathbb{U} \rangle] = v\left[\sum_{i=1}^{3} u^{i^2}\right]$$

$$= \sum_{i=1}^{3} 2u^i v[u^i] = 2\langle \mathbb{U}, \nabla_v \mathbb{U} \rangle$$

So $\nabla_v \mathbb{U} \in T_pM$.

(2) S_p is linear

$\forall \ v, w \in T_pM, \quad a \in \mathbb{R}$

$$S_p(av+w) = -\nabla_{av+w} U$$

$$= -(av+w)(u^i)e_i$$

$$= -\left(av(u^i)e_i + w(u^i)e_i\right)$$

$$= -a\nabla_v U - \nabla_w U$$

$$= aS_p(v) + S_p(w) \quad . \qquad \#$$

Ex: Try to compute the shape operator of $S^2(R)$ at any pt p.

One may consider the parametrization (covers most of $S^2(R)$)

$$X(u,v) = (R\cos u\cos v, R\sin u\cos v, R\sin v)$$

$$\Rightarrow \begin{cases} X_u = R(-\sin u\cos v, \cos u\cos v, 0) \\ X_v = R(-\cos u\sin v, -\sin u\sin v, \cos v) \end{cases}$$

$$\Rightarrow \quad X_u \times X_v = R^2\cos v(\cos u\cos v, \sin u\cos v, \sin v)$$

$$\Rightarrow U = (\cos u\cos v, \sin u\cos v, \sin v)$$

$$S_p(X_u) = -\nabla_{X_u} U = -\sum_i X_u[u^i]\, e_i$$

$$= -(-\sin u \cdot \cos v,\ \cos u \cdot \cos v,\ 0)$$

$$= -\frac{1}{R} X_u$$

$$S_p(X_v) = -\frac{1}{R} X_v$$

Therefore $\forall\ v = \lambda X_u + \mu X_v \in T_pM$

$$S_p(v) = -\frac{1}{R} v$$

$$(\text{i.e } S_p = -\frac{1}{R} \mathrm{Id}_{T_pM})$$