Tutorial 3 19/09/2014

① Let's recall the def. of surfaces (in $\mathbb{R}^3$).

Let $D \subset \mathbb{R}^2$ = open connected set

$X : D \to \mathbb{R}^3$ map

$(u,v) \mapsto (x^1(u,v), x^2(u,v), x^3(u,v))$

$$X_u = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u} \right)$$

$$X_v = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v} \right)$$

We call X is regular if $X_u \times X_v \neq 0 \quad \forall (u,v) \in D$.

A coordinate patch (or parametrization) is a 1-1, regular mapping $X : D \to \mathbb{R}^3$ where D is a connected open set in $\mathbb{R}^2$.

A surface M in $\mathbb{R}^3$ is a subset in $\mathbb{R}^3$ s.t $\forall x \in M$, $\exists$ neighborhood of x, $U (\subset M)$ is covered by some coordinate patch i.e $X(D) \supset U$.

A surface M is differentiable (or smooth) if $\forall$ coordinate patch $X : D_1 \to M$, $Y : D_2 \to M$ with $X(D_1) \cap Y(D_2) \neq \phi$

the composition $X^{-1} \circ Y : \underbrace{Y^{-1}(X(D_1) \cap Y(D_2))}_{\subset \mathbb{R}^2} \to \underbrace{X^{-1}(X(D_1) \cap Y(D_2))}_{\subset \mathbb{R}^2}$

is differentiable. (i.e changes of coordinates are differentiable.

Ex: $X(u,v) = \big((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u \big)$

$(R>r>0$ and $u,v \in [-\pi,\pi))$

Show that X is a coordinate patch.

Pf: Regular: $X_u = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u)$

$X_v = (-(R+r\cos u)\sin v,\ (R+r\cos u)\cos v,\ 0)$

$$X_u \times X_v = \begin{vmatrix} i & j & k \\ -r\sin u\cos v & -r\sin u\sin v & r\cos u \\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0 \end{vmatrix}$$

$= \big(-r(R+r\cos u)\cos u\cos v,\ -r(R+r\cos u)\cos u\sin v,\ -r(R+r\cos u)\sin$

Since $R>r>0$, so if $u \neq -\pi$ or 0

$X_u \times X_v \neq 0$ (Consider z)

If $u=-\pi$, $X_u \times X_v = (r(R-r)\cos v,\ r(R-r)\sin v,\ 0) \neq 0$

If $u=0$, $X_u \times X_v = (-r(R+r)\cos v,\ -r(R+r)\sin v,\ 0) \neq 0$.

1-1: Now we have

$$\begin{cases} (R+r\cos u_1)\cos v_1 = (R+r\cos u_2)\cos v_2 & \text{①} \\ (R+r\cos u_1)\sin v_1 = (R+r\cos u_2)\sin v_2 & \text{②} \\ r\sin u_1 = r\sin u_2 & \text{③} \end{cases}$$

①^2 + ②^2 :

$$(R+r\cos u_1)^2 = (R+r\cos u_2)^2$$

$$\Rightarrow \quad R+r\cos u_1 = R+r\cos u_2$$

$$\Rightarrow \quad \cos u_1 = \cos u_2$$

Combine with ③

$$\Rightarrow \quad u_1 = u_2$$

by ①②

$$\Rightarrow \quad \begin{cases} \cos v_1 = \cos v_2 \\ \sin v_1 = \sin v_2 \end{cases}$$

$$\Rightarrow \quad v_1 = v_2$$

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Ex: Show if $f: \mathbb{R}^3 \to \mathbb{R}$ (C^∞), then when restricting to a surface M

$$X_u(X_v(f)) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} \cdot \frac{\partial x^i}{\partial u} \cdot \frac{\partial x^j}{\partial v} + \sum_i \frac{\partial f}{\partial x^i} \cdot \frac{\partial^2 x^i}{\partial v \partial u}$$

Where $X(u,v)$ is a coordinate patch.

Pf: $X_v(f) = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial v}$

$$X_u(X_v f)) = \frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v} + \frac{\partial f}{\partial x^i}\frac{\partial^2 x^i}{\partial u \partial v}.$$

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Ex 2.3.12 in Oprea's book:

Compute the Gauss map for Enneper's Surface $X(u,v) = (u - u^3/3 + uv^2, v - v^3/3 + vu^2, u^2 - v^2)$ and show that it's a 1-1 map from Enneper's surface to the sphere. Also show that the image of the disk $\{(u,v) \mid u^2+v^2 \leq 3\}$ under G covers more than a hemisphere of the sphere.

Pf: Gauss map $\left(\frac{X_u \times X_v}{|X_u \times X_v|}\right)$

$X_u = (1-u^2+v^2, 2uv, 2u)$

$X_v = (2uv, 1-v^2+u^2, -2v)$

$X_u \times X_v = (-2u(u^2+v^2+1), 2v(u^2+v^2+1), 1-(u^2+v^2)^2)$

Change to polar coordinate $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$

$$X_u \times X_v = \left(-2r\cos\theta\,(r^2+1),\ 2r\sin\theta\,(r^2+1),\ 1-r^4\right)$$

$$\Rightarrow |X_u \times X_v| = (r^2+1)^2$$

$$\Rightarrow G = \left(\frac{-2r\cos\theta}{r^2+1},\ \frac{2r\cos\theta}{r^2+1},\ \frac{1-r^2}{1+r^2}\right)$$

Check 1-1:

$$\begin{cases} \dfrac{-2r_1\cos\theta_1}{r_1^2+1} = \dfrac{-2r_2\cos\theta_2}{r_2^2+1} & (1) \\[2ex] \dfrac{2r_1\sin\theta_1}{r_1^2+1} = \dfrac{2r_2\sin\theta_2}{r_2^2+1} & (2) \\[2ex] \dfrac{1-r_1^2}{1+r_1^2} = \dfrac{1-r_2^2}{1+r_2^2} & (3) \end{cases}$$

By (3), $1-r_1^2+r_2^2-r_1^2r_2^2 = 1-r_1^2r_2^2-r_2^2+r_1^2$

$\Rightarrow r_1^2 = r_2^2$

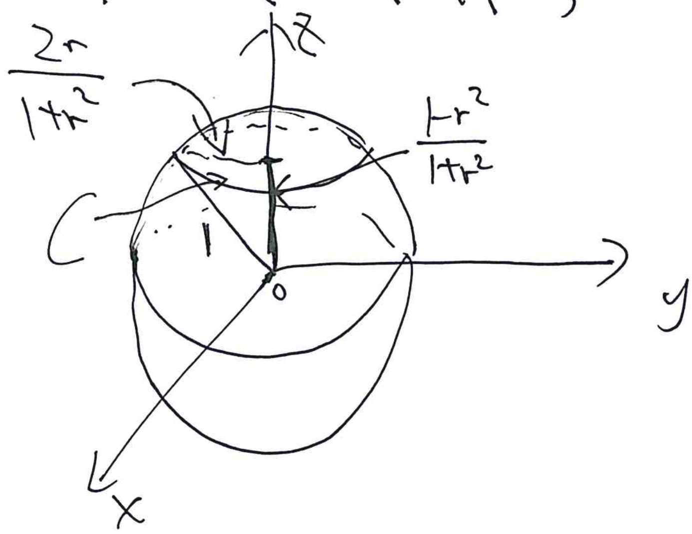
Since $r_1, r_2 \geq 0 \Rightarrow r_1 = r_2$

Plug it in (1), (2), $\begin{cases} \cos\theta_1 = \cos\theta_2 \\ \sin\theta_1 = \sin\theta_2 \end{cases}$

$\Rightarrow \theta_1 = \theta_2$.

Now we show the image of $\{(u,v) \mid u^2+v^2 \le 3\}$ under G covers more than a hemisphere.

$$G = \left(\frac{-2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{1-r^2}{1+r^2} \right)$$



Any pt (x_0, y_0, z_0) in the upper hemisphere, we can find (r_0, θ_0) s.t

$$\begin{cases} x_0 = -\dfrac{2r_0\cos\theta_0}{1+r_0^2} & ① \\ y_0 = \dfrac{2r_0\sin\theta_0}{1+r_0^2} & ② \\ z_0 = \dfrac{1-r_0^2}{1+r_0^2} & ③ \end{cases}$$

where $r_0 \in [0,3]$, $\theta \in [0, 2\pi)$.

$z(r) = \dfrac{1-r^2}{1+r^2} \searrow [0,1]$ $\quad z(0)=1$

$z(1) = 0$.

Given $z_0 \in [0,1]$, $\exists! r_0 \in [0,1]$ s.t ③.

Any pt in C can be represented by ① and ②.

And if $r > 1$, then $z < 0$. We get the desired result. #

Ex: $X(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$

Show that X is regular $\Leftrightarrow$ $\alpha(u) = (g(u), h(u), 0)$ is regular and has no intersection with X-~~aix~~axis.

1 of Ex in Tutorial 3 :

($\Rightarrow$) We compute $\begin{cases} X_u = (g', h'\cos v, h'\sin v) \\ X_v = (0, -h\sin v, h\cos v) \end{cases}$

$$\Rightarrow \quad X_u \times X_v = (h'h, -g'h\cos v, -g'h\sin v)$$

And compute $\quad \alpha' = (g', h', 0)$

If $g' = h' = 0$, then $X_u \times X_v = (0,0,0)$ $\rightarrow\leftarrow$.

If $h = 0$, then $X_u \times X_v = (0,0,0)$ $\rightarrow\leftarrow$.

So $\alpha(u) = (g(u), h(u), 0)$ is regular and has no intersection with x-axis.

($\Leftarrow$) Assume ~~we have~~ ~~$g' \neq 0$~~ ~~then~~ α is regular and has no intersection with x-axis. We have

$$h \neq 0 \quad \text{and} \quad g', h' \text{ can't equal to } 0 \text{ at the same time.}$$

If $h' = 0$, $X_u \times X_v = (0, -g'h\cos v, -g'h\sin v) \neq 0$

If $g' = 0$, $X_u \times X_v = (h'h, \quad " \quad , \quad " \quad) \neq 0$

If $h' \neq 0$ and $g' \neq 0$, $X_u \times X_v \neq 0$.

So X is regular.

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