Tutorial 2 12/09/2014

① Isoperimetric inequality : A method by the theory of Fourier

Recall : • A closed (Smooth) plane curve is a parametrized curve $\alpha : [a,b] \to \mathbb{R}^2$ s.t $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ $\forall n \geq 0$

where $\alpha^{(n)}$ = n-th derivative of α.

• A curve is simple if it has no self-intersection.

Isoperimetric ineq.: Let C be a smooth closed simple plane curve with Length L enclosing an area A. Then

$L^2 \geq 4\pi A$. Moreover, "=" holds iff

C = a circle.

Wirtinger's Lemma: If $f(t)$ is a cts, periodic function with period 2π and $f'(t)$ is cts. What's more, $\int_0^{2\pi} f(t)dt =$

Then
$$\int_0^{2\pi} [f'(t)]^2 dt \geq \int_0^{2\pi} [f(t)]^2 dt.$$

And "=" holds iff $f(t) = a\cos t + b\sin t$ where a, b are some constants.

Let's assume the above lemma is true. (We will show it later.)

Pf of thm :

Assume $C = (x(s), y(s))$, $s \in [0, L]$, s = arc-length

Consider $t \triangleq \frac{2\pi}{L} s$, $t \in [0, 2\pi]$. $\begin{cases} \tilde{x}(t) = x(s(t)) \\ \tilde{y}(t) = y(s(t)) \end{cases}$

One can choose a coordinate system s.t $\int_0^{2\pi} \tilde{x}(t)\, dt = 0$

Since $\int_0^{2\pi} \tilde{x}(t)\, dt = a$, then $\int_0^{2\pi} \left(\tilde{x}(t) - \frac{a}{2\pi}\right) dt = 0$.

For convenience, $\tilde{x} = x$.

Note that we have

$$x'^2 + y'^2 = (\dot{x}^2 + \dot{y}^2)\left(\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \frac{L^2}{4\pi^2}$$

Here " $'$ " denotes the derivative w.r.t t and " $\cdot$ " denotes the derivative w.r.t s.

$$\Rightarrow \quad \frac{L^2}{2\pi} = \int_0^{2\pi} (x'^2 + y'^2)\, dt \qquad (1)$$

By Green's Thm,
$$A = \int_C x\, dy = \int_0^{2\pi} x y'\, dt \qquad (2)$$

[Recall Green's Thm: Let R = simply connected region with boundary curve C in $\mathbb{R}^2$, $P(x,y)$, $Q(x,y)$ = C^∞ functi on R. Then $\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx dy = \int_C P dy - Q dx$.

Corol. $\text{Area}(R) = \frac{1}{2}\int_C x dy - y dx = \int_C x dy = \int_C -y dx$

(Pf: Take $P = \frac{x}{2}$, $Q = \frac{y}{2}$ to get 1st equ.

$P = x$, $Q = 0$ ..

$P = 0$, $Q = y$ ..)]

By (1) and (2), we have

$$2\left(\frac{L^2}{4\pi} - A\right) = \int_0^{2\pi} (x'^2 + y'^2 - 2xy')\, dt$$

$$= \int_0^{2\pi} (x'^2 - x^2) dt + \int_0^{2\pi} (x - y')^2 dt$$

$$\geq \int_0^{2\pi} (x'^2 - x^2) dt$$

≥ 0 by Wirtinger's Lemma applying to $x(t)$

So we have $\frac{L^2}{4\pi} - A \geq 0$.

"=" holds iff $x = a\cos t + b\sin t$

$$y' = x$$

$$\Downarrow$$

$$y = a\sin t - b\cos t + c$$

$$\Rightarrow \quad x^2 + (y-c)^2 = a^2 + b^2$$

On the other hand, $C =$ a circle $\Rightarrow L^2 = 4\pi A$.

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Now let's show the Wirtinger's lemma:

We see that $f(t)$ can be represented by its Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt)$$

Where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t)\,dt$

$$\begin{cases} a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt \\ b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt. \end{cases}$$

[Question: What kind of function can be represented by its Fourier series?]

By the assumption, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = 0$.

(Check

Since $f'(t)$ is cts, then we can take derivative term by term.

$$\Rightarrow \quad f'(t) = \sum_{k=1}^{+\infty} k\,(b_k \cos kt - a_k \sin kt)$$

Apply the Parseval's Thm to $f(t)$ and $f'(t)$, we have

$$\frac{1}{\pi}\int_0^{2\pi} [f(t)]^2\,dt = \sum_{k=1}^{+\infty} (a_k^2 + b_k^2) \qquad (3)$$

$$\frac{1}{\pi}\int_0^{2\pi} [f'(t)]^2\,dt = \sum_{k=1}^{+\infty} k^2 (a_k^2 + b_k^2) \qquad (4)$$

(Parseval's Thm: (A special case)

$$\frac{1}{\pi}\int_0^{2\pi} f^2(t)\,dt = \frac{a_0^2}{2} + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2)$$

whenever $f \in L^2([0,2\pi])$.)

(4) − (3), we have

$$\int_0^{2\pi} f'^2\,dt - \int_0^{2\pi} f^2\,dt = \pi \sum_{k=1}^{+\infty} (k^2-1)(a_k^2 + b_k^2) \geq 0.$$

"=" holds iff $a_k = b_k = 0$, $\forall k \geq 2$.

i.e

$$f(t) = a_1 \cos t + b_1 \sin t.$$

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(2) Ex: Let $\alpha(s)$ be a unit speed curve with curvature $\kappa(s)>0$ and Frenet frame $\{T, N, B\}$. Suppose $w(s)$ is a vector field along α s.t. $T' = w\times T$, $N' = w\times N$, $B' = w\times B$ $\forall s$. Show that $w = \tau T + \kappa B$, where τ is the torsion of α and $T'\times T'' = \kappa^2 w$.

Pf: Assume $w = aT + bN + cB$

$$\begin{cases} T' = w\times T \\ N' = w\times N \\ B' = w\times B \end{cases} \Rightarrow \begin{cases} bN\times T + cB\times T = T' \overset{\text{Frenet Formula}}{=} \kappa N & (1) \\ aT\times N + cB\times N = N' = -\kappa T + \tau B & (2) \\ aT\times B + bN\times B = B' = -\tau N & (3) \end{cases}$$

(1) inner products with N, $c = \kappa$

(3) . . . . T, $b = 0$

(3) . . . N, $-a = -\tau$

$a = \tau$.

$\Rightarrow$ $w = \tau T + \kappa B$.

$T' = \kappa N$ $\quad T'' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B)$

$T'\times T'' = -\kappa^3 N\times T + \kappa\tau N\times B$

$= +\kappa^3 B + \kappa\tau T$

$$\kappa^2 W = \kappa^2 \tau T + \kappa^3 B.$$

So $T' \times T'' = \kappa^2 W$.

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