Tutorial 1 : 12/09/2013

① Local behavior of a curve :

Let $\alpha(s) = (\alpha^1(s), \alpha^2(s), \alpha^3(s))$ be a unit speed curve i.e parametrized by arc-length. Assume $k(0) > 0$.
(Recall: the curvature k is defined by $|\alpha''(s)|$)

By Frenet's formula,

$$\alpha' = T$$

$$\alpha'' = T' = kN$$

$$\left(\begin{cases} T' = & kN & \\ N' = & -kT & +\tau B \\ B' = & -\tau N & \end{cases} \quad \text{where } \tau = -\langle B', N \rangle . \right)$$

$$\alpha''' = k'N + kN' = k'N + k(-kT + \tau B)$$

$$= -k^2 T + k'N + k\tau B .$$

We may choose the coordinate system of $\mathbb{R}^3$ s.t

$$\begin{cases} \alpha(0) = (0, 0, 0) \\ T(0) = e_1 , \ N(0) = e_2 , \ B(0) = e_3 \end{cases}$$

By Taylor expansion,

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + \alpha'''(0)\frac{s^3}{6} + o(s^3)$$

where $o(s^3)$ means $\lim_{s \to 0} \frac{\text{the remainder terms}}{s^3} = 0$.

$$= sT(0) + \frac{s^2}{2}\kappa(0)N(0) + \frac{s^3}{6}\left(-\kappa^2(0)T(0) + \kappa'(0)N(0) + \kappa(0)\tau(0)B(0)\right)$$
$$+\ o(s^3)$$

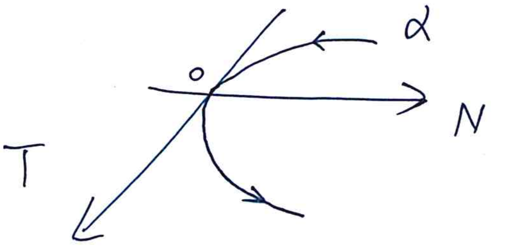
$$= \left(s - \frac{\kappa^2(0)}{6}s^3 + o(s^3)\right)e_1 + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3 + o(s^3)\right)e_2$$
$$+ \left(\frac{\kappa(0)\tau(0)}{6}s^3 + o(s^3)\right)e_3$$

i.e.

$$\begin{cases} x = \alpha^1(s) = s - \dfrac{\kappa^2(0)}{6}s^3 + o(s^3) \\ y = \alpha^2(s) = \dfrac{\kappa(0)}{2}s^2 + \dfrac{\kappa'(0)}{6}s^3 + o(s^3) \\ z = \alpha^3(s) = \dfrac{\kappa(0)\tau(0)}{6}s^3 + o(s^3) \end{cases}$$

Consider the projection of α on TN-Plane :

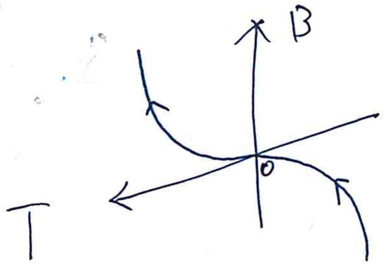
$$y = \frac{\kappa(0)}{2}x^2 + \cdots \qquad (\text{Parabola})$$



$(\kappa(0) > 0)$

Consider the projection of α on TB-Plane :

$$z = \frac{\kappa(0)\tau(0)}{6}x^3 + \cdots$$



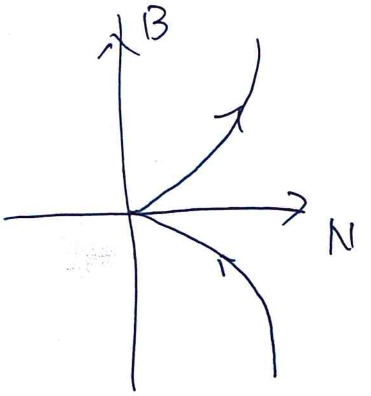
(Picture shows the case $\tau(0) > 0$)

Consider the projection on NB - Plane :

Like a cusp $\sim$ (t^2, t^3) ← $\tau(0) > 0$

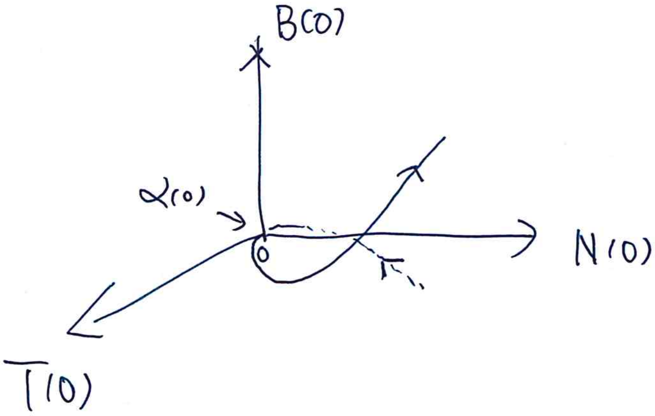
or

$(t^2, -t^3)$ ← $\tau(0) < 0$



(Picture shows the case $\tau(0) > 0$)

$\Rightarrow$ α (at $s = 0$) looks like



(Picture shows the case $\tau(0) > 0$)

(2) Ex: α unit speed. Assume $\tau \neq 0$, $\kappa \neq 0$, $\kappa' \neq 0$.

α lies on a sphere $\Leftrightarrow$ $\frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau} \right)^2 = \text{constant}$.

Pf: ($\Rightarrow$) Assume α lies on a sphere

i.e $\exists\, p \in \mathbb{R}^3$ s.t

$$|\alpha - p|^2 = \text{Constant}$$

We may assume $p = 0$, since if we change the coordinate system of $\mathbb{R}^3$, there is no change of κ and τ.

Thus $|\alpha|^2 = \text{constant}$.

Since $|\alpha|^2 = \langle \alpha, B \rangle^2 + \langle \alpha, T \rangle^2 + \langle \alpha, N \rangle^2$, this inspires us to compute $\langle \alpha, B \rangle$, $\langle \alpha, T \rangle$, $\langle \alpha, N \rangle$.

$$|\alpha|^2 = c \Rightarrow \langle \alpha', \alpha \rangle = 0 \Rightarrow \langle \alpha'', \alpha \rangle + \langle \alpha', \alpha' \rangle = 0$$

$$\Rightarrow \langle \alpha'', \alpha \rangle = -1 \Rightarrow \langle \alpha''', \alpha \rangle + \cancel{\langle \alpha'', \alpha' \rangle} = 0$$

Note that

$$\alpha''' = k'N - k^2T + k\tau B$$

$$\alpha'' = kN$$

$\therefore \quad \langle \alpha'', \alpha \rangle = -1 \Rightarrow \quad k\langle \alpha, N \rangle = -1$

$\Rightarrow \quad \langle \alpha, N \rangle = -\frac{1}{k}$.

$\langle \alpha''', \alpha \rangle = 0 \Rightarrow k'\langle \alpha, N \rangle - k^2\langle \alpha, T \rangle + k\tau\langle \alpha, B \rangle = 0$

$\langle \alpha', \alpha \rangle = 0 \Rightarrow \langle \alpha, T \rangle = 0$.

$\Rightarrow \quad \langle \alpha, B \rangle = -\frac{k'}{k\tau}\langle \alpha, N \rangle = \frac{k'}{k^2\tau} = -\left(\frac{1}{k}\right)' \cdot \frac{1}{\tau}$

$\therefore \quad \left(\frac{1}{k}\right)^2 + \left[\left(\frac{1}{k}\right)'\frac{1}{\tau}\right]^2 = \text{Constant}.$

$(\Leftarrow)$ Assume $\left(\frac{1}{k}\right)^2 + \left[\left(\frac{1}{k}\right)'\frac{1}{\tau}\right]^2 = \text{Constant}$,

how can we find a sphere s.t α lies in it ?

From "($*$ $\Rightarrow$)", we may consider $\alpha(s) - p(s) = -\frac{1}{k}N - \left(\frac{1}{k}\right)'\frac{1}{\tau}B$

i.e $p(s) = \alpha + \frac{1}{k}N + \left(\frac{1}{k}\right)'\frac{1}{\tau}B$

If we can show $p' = 0$, then we are done.

Now we show $p' = 0$.

$$p' = \alpha' + \left(\tfrac{1}{\kappa}\right)'N + \tfrac{1}{\kappa}\overset{-\kappa T + \tau B}{N'} + \left(\tfrac{1}{\kappa}\right)''\tfrac{1}{\tau}B + \left(\tfrac{1}{\kappa}\right)'\left(\tfrac{1}{\tau}\right)'B + \left(\tfrac{1}{\kappa}\right)'\tfrac{1}{\tau}\overset{-\tau N}{B'}$$

$$= B\left(\frac{\tau}{\kappa} + \left(\tfrac{1}{\kappa}\right)''\tfrac{1}{\tau} + \left(\tfrac{1}{\kappa}\right)'\left(\tfrac{1}{\tau}\right)'\right)$$

$$= B\,\frac{\left[\left(\tfrac{1}{\kappa}\right)^2 + \left[\left(\tfrac{1}{\kappa}\right)'\tfrac{1}{\tau}\right]^2\right]'}{2\left(\tfrac{1}{\kappa}\right)'\left(\tfrac{1}{\tau}\right)}$$

Here we use $\left(\frac{1}{\kappa}\right)' \neq 0$
$\Leftrightarrow \kappa' \neq 0$.

$$= 0.$$

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①$(\alpha''(0) \neq 0)$ ②$(\alpha''(0) = 0)$

Ex: Let $\alpha(s)$ be a unit speed curve defined in a neighborhood of $s = 0$. (i) Show that there exists a unique circle $\beta(s)$ s.t.

$$\beta(0) = \alpha(0), \quad \beta'(0) = \alpha'(0), \quad \beta''(0) = \alpha''(0)$$

(ii) What's the radius of the circle β?

Sol of Ex in Tutorial 1 : ① $\alpha''(0) \neq 0$.

We may use the frame $\{ o \mid e_1, e_2, e_3 \}$ s.t

$$\alpha(0) = 0, \quad \alpha'(0) = e_1, \quad \frac{\alpha''(0)}{|\alpha''(0)|} = e_2$$

$$e_3 = e_1 \times e_2 .$$

One may guess the centre of the circle is

$$c_0 = \frac{1}{|\alpha''(0)|} e_2$$

$$r = \frac{1}{|\alpha''(0)|}$$

Then we ~~may~~ can write down the circle in this way:

$$\beta(s) = c_0 + \left(r \sin\frac{s}{r}, -r\cos\frac{s}{r}, 0 \right)$$

One can easily check

$$\beta(0) = 0, \quad \beta'(0) = e_1 = \alpha'(0), \quad \beta''(0) = \frac{1}{r} e_2 = \alpha''(0).$$

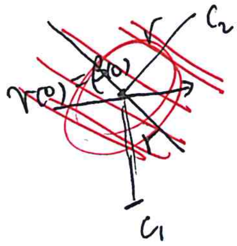
This gives the existence. Consider another circle $\gamma(s)$ s.t

$$\gamma(0) = \beta(0), \quad \gamma'(0) = \beta'(0), \quad \gamma''(0) = \beta''(0)$$

$\downarrow$ through the same pt

$\downarrow$ gives the same radius (Same direct.

and in the same plane



~~Finally, $\gamma'(0) = \beta'(0)$ gives they are the same actually~~. This shows the uniqueness.

radius of $\beta = \frac{1}{|\alpha''(0)|}$.

② $\alpha''(0) = 0$. Consider $\beta(s) = se_1$ (a line)

Which can be regard as a circle with infinite radius.

Similar argument works.

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