Tutorial 9 31/10/2014

① An Ex. in last tutorial :

$$H = \frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)\, d\theta \quad , \quad k_n(\theta) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$$

$$= \frac{1}{2\pi} k_1 \int_0^{2\pi} \cos^2\theta\, d\theta + \frac{1}{2\pi} k_2 \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{1}{2\pi} k_1 \int_0^{2\pi} \frac{\cos 2\theta + 1}{2}\, d\theta + \frac{1}{2\pi} k_2 \int_0^{2\pi} \frac{-\cos 2\theta + 1}{2}\, d\theta$$

$$= \frac{1}{2\pi} k_1 \cdot \pi + \frac{1}{2\pi} k_2 (+\pi)$$

$$= \frac{1}{2}(k_1 + k_2).$$

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② Thm: A surface M consisting entirely of umbilic pts is contained in either a plane or a sphere.

(Recall umbilic pt : $k_1 = k_2$)

Pf: Umbilic at all pts $\Rightarrow$ $k_1(p) = k_2(p) \quad \forall p \in M$

$(= k(p) = \text{a } C^\infty \text{ function on } M)$

$$\Rightarrow \begin{cases} \nabla_{X_1} U = -S(X_1) = -k X_1 \\ \nabla_{X_2} U = -S(X_2) = -k X_2 \end{cases} \qquad \forall p \in M$$

$$\Rightarrow \begin{cases} \nabla_{X_2}\nabla_{X_1} U = -(\nabla_{X_2} k)\cdot X_1 - k \nabla_{X_2} X_1 \\ \nabla_{X_1}\nabla_{X_2} U = -(\nabla_{X_1} k)\cdot X_2 - k \nabla_{X_1} X_2 \end{cases}$$

Note that

$$\begin{cases} \nabla_{X_2} X_1 = X_{12} = \nabla_{X_1} X_2 \\ \nabla_{X_1}\nabla_{X_2} U = \nabla_{X_2}\nabla_{X_1} U \end{cases}$$

$$\Rightarrow \quad (\nabla_{X_2} k)\cdot X_1 = (\nabla_{X_1} k)\cdot X_2$$

$$\Rightarrow \quad \nabla_{X_2} k = \nabla_{X_1} k = 0$$

$\Rightarrow$ k is a constant on M.

Case (a) : If $k = 0$, then $S_p \equiv 0$ $\forall p \in M$.

$\Rightarrow$ M is contained in a plane.

Case (b) : If $k \neq 0$, then consider

$$X + \frac{1}{k} U.$$

$$\frac{\partial}{\partial u_i}\left(X + \frac{1}{k}U\right) = X_i + \frac{1}{k}\nabla_{X_i} U$$

$$= X_i - \frac{1}{k} S(X_i)$$

$$= X_i - \frac{1}{k} k X_i$$

$$= 0. \qquad \forall i = 1,2$$

$$\Rightarrow \quad X + \frac{1}{k} U = P_0$$

$\Rightarrow$ $|X - P_0|^2 = \frac{1}{k^2}$ $\Rightarrow$ M is contained in a sphere.

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③ Thm: $M =$ cpt surface in $\mathbb{R}^3$ with constant Gauss curvature K

$\Rightarrow$ M is a sphere with radius $\frac{1}{\sqrt{K}}$.

Lemma: $M =$ cpt surface in $\mathbb{R}^3$

$p \in M$ s.t $k_1(p) = \max\limits_M k_1 > k_2(p) = \min\limits_M k_2$

$\Rightarrow$ $K(p) \leq 0$.

We first use the above lemma to show the Thm.

By a thm in last tutorial, M cpt $\Rightarrow$ $\exists\, p_0 \in M$ s.t $K(p_0) >$

$\Rightarrow$ K is a positive constant.

M is cpt + $k_1(p) \cdot k_2(p) \equiv K$ $\forall p \in M$.

$$\Rightarrow \quad \exists\, p \in M \text{ s.t } \begin{cases} k_1(p) = \max\limits_M k_1 \\ k_2(p) = \min\limits_M k_2 \end{cases}$$

If $k_1(p) > k_2(p)$, by the above lemma $\Rightarrow$ $K(p) \leq 0$ $\rightarrow\leftarrow$.

So $k_1(p) = k_2(p)$ $\Rightarrow$ M is umbilic at all pts.

$\Rightarrow$ M is contained in a sphere $\xRightarrow{+\text{ cpt}}$ Sphere

with radius $\frac{1}{\sqrt{K}}$ $\left(= \frac{1}{k_1} = \frac{1}{k_2}\right)$ #

Pf of the Lemma:

By continuity, $k_1 > k_2$ in a neighborhood of p, Ω°. Then
$\exists$ a coordinate patch at a smaller neighborhood of p $\Omega' \subsetneq \Omega$ s.t.

$$\begin{cases} S(X_1) = k_1 X_1 \\ S(X_2) = k_2 X_2 \end{cases}$$

(Ex: Why can we find such a neighborhood?)

Since $k_1 > k_2 \Rightarrow \langle X_1, X_2 \rangle = 0$ i.e $g_{12} = 0$ on Ω'.

Then
$$\begin{cases} h_{11} = \langle S(X_1), X_1 \rangle = k_1 g_{11} \\ h_{12} = \langle S(X_1), X_2 \rangle = 0 \\ h_{22} = \langle S(X_2), X_2 \rangle = k_2 g_{22} \end{cases}$$

i.e
$$g = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \quad h = \begin{pmatrix} k_1 g_{11} & 0 \\ 0 & k_2 g_{22} \end{pmatrix} \quad \text{on } \Omega'.$$

Recall the Codazzi equations:

$$\frac{\partial h_{ij}}{\partial u^k} + \sum_{s=1}^{2} h_{ks} \Gamma_{ij}^s = \frac{\partial h_{ik}}{\partial u^j} + \sum_{s=1}^{2} h_{js} \Gamma_{ik}^s \quad \forall\, i,j,k = 1,2$$

$$\left(\Gamma_{ij}^k = \tfrac{1}{2} g^{kl} \left(g_{jl,i} + g_{il,j} - g_{ij,l} \right) \right)$$

Take $i = j = 1$, $k = 2$, we have

$$h_{11,2} + h_{22} \Gamma_{11}^2 = h_{11} \Gamma_{12}^1$$

i.e
$$\frac{\partial (k_1 g_{11})}{\partial u^2} + k_2 g_{22} \Gamma_{11}^2 = k_1 g_{11} \Gamma_{12}^1$$

$$g_{12}=0 \Rightarrow \Gamma_{11}^{2} = \frac{1}{2} g^{22}(-g_{11,2}) = -\frac{1}{2}\frac{1}{g_{22}}\frac{\partial g_{11}}{\partial u_2}$$

$$\Gamma_{12}^{1} = \frac{1}{2} g^{11}\cdot\frac{\partial g_{11}}{\partial u_2} = \frac{1}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial u_2}$$

$$\Rightarrow \quad k_1 g_{11,2} + \frac{\partial k_1}{\partial u^2}\cdot g_{11} = \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} k_2 + \frac{1}{2} k_1 \frac{\partial g_{11}}{\partial u_2}$$

$$\Rightarrow \quad \frac{\partial k_1}{\partial u^2} = \frac{1}{2}\cdot\frac{(k_2-k_1)}{g_{11}} g_{11,2} \qquad (1)$$

If we take $i=j=2$, $k=1$ in the Codazzi equations, we have

$$\frac{\partial k_2}{\partial u^1} = \frac{1}{2}\frac{(k_1-k_2)}{g_{22}} g_{22,1}. \qquad (2)$$

Diff. (1) and (2) again:

$$\begin{cases} \dfrac{\partial^2 k_1}{\partial (u^2)^2} = \dfrac{k_2-k_1}{2g_{11}}\dfrac{\partial^2 g_{11}}{\partial (u^2)^2} + (\ldots)\dfrac{\partial g_{11}}{\partial u^2} & (3) \\[2ex] \dfrac{\partial^2 k_2}{\partial (u^1)^2} = \dfrac{k_1-k_2}{2g_{22}}\dfrac{\partial^2 g_{11}}{\partial (u^1)^2} + (\ldots)\dfrac{\partial g_{22}}{\partial u^1} & (4) \end{cases}$$

Since $k_1(p) = \max k_1$ and $k_2(p) = \min k_2$ (p is an interior of Ω)

we have

$$(*) \quad \frac{\partial k_1}{\partial u^2}(p) = \frac{\partial k_2}{\partial u^2}(p) = 0,$$

$$\frac{\partial^2 k_1}{\partial (u^2)^2}(p) \leq 0, \quad \frac{\partial^2 k_2}{\partial (u^1)^2}(p) \geq 0.$$

Put $(*)$ into (1) (2) (3) (4), we have at p

$$\frac{\partial g_{11}}{\partial u^2} = \frac{\partial g_{22}}{\partial u^1} = 0 \quad \text{and}$$

$$\frac{\partial^2 g_{11}}{\partial (u^2)^2}, \; \frac{\partial^2 g_{22}}{\partial (u^1)^2} \geq 0.$$

By a special formula ($g_{12} \equiv 0$) of Gauss curvature,

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left(\frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u^2}\right) + \frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u^1}\right)\right)$$

$$= -\frac{1}{2\sqrt{g_{11}g_{22}}}\left((\ldots)\cancel{\frac{\partial g_{11}}{\partial u^2}}^{\,0} + \frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial^2 g_{11}}{\partial (u^2)^2} + (\ldots)\cancel{\frac{\partial g_{22}}{\partial u^1}} + \frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial^2 g_{22}}{\partial (u^1)^2}\right)$$

≤ 0 at p.

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Ex: Let $X(u,v) = (u, v, g(u) + h(v))$ be a minimal surface ($H \equiv 0$). Show that

$$(1 + h'^2)g'' + [1 + g'^2]h'' = 0.$$

Try to find $f(u,v)$ explicitly s.t $f(u,v) = g(u) + h(v)$.