Tutorial 8 24/10/2014

① Ex in last tutorial : Show that the Gauss curvature of a ruled surface $X(u,v) = \beta(u) + v\delta(u)$ is

$$K = -\frac{\langle \beta', \delta \times \delta' \rangle^2}{|\beta' \times \delta + v\delta' \times \delta|^4}.$$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{\begin{vmatrix} l & m \\ m & n \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

$E = \langle X_u, X_u \rangle$, $F = \langle X_u, X_v \rangle$

$G = \langle X_v, X_v \rangle$

$l = \langle X_{uu}, U \rangle$, $m = \langle X_{uv}, U \rangle$

$n = \langle X_{vv}, U \rangle$.

$X_u = \beta' + v\delta'$, $X_v = \delta$, $X_{uu} = \beta'' + v\delta''$,

$X_{uv} = \delta'$, $X_{vv} = 0$.

parallelepi

$$X_u \times X_v = \beta' \times \delta + v\delta' \times \delta, \quad U = \frac{\beta' \times \delta + v\delta' \times \delta}{|\beta' \times \delta + v\delta' \times \delta|}$$

$$\det \mathrm{I} = |X_u \times X_v|^2 = (EG - F^2) = |\beta' \times \delta + v\delta' \times \delta|^2$$

$$l = \frac{1}{|\beta' \times \delta + v\delta' \times \delta|}\Big(\langle \beta'', \beta' \times \delta \rangle + v\langle \delta'', \beta' \times \delta \rangle + v\langle \beta'', \delta' \times \delta \rangle + v^2 \langle \delta'', \delta' \times \delta \rangle \Big)$$

$$m = \langle \delta', \beta' \times \delta \rangle \frac{1}{|\cdots|} \qquad K = -\frac{\langle \beta', \delta \times \delta' \rangle^2}{|\cdots|^4}.$$

$n = 0$

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② Let M be a regular surface. And let $\lambda_1, \cdots, \lambda_m$, $m > 2$ be the normal curvatures at $p \in M$ along directions making angle $0, 2\pi/m, \cdots, (m-1)2\pi/m$ with a principal direction. Prove that $\lambda_1 + \cdots + \lambda_m = mH$, where H is the mean curvature at

$(k_n(v) = \langle S_p(v), v \rangle)$.

Pf: By Euler's formula,

$$k_n(v) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$$

Where θ is the angle from v_1 (a unit ~~[illegible]~~ principal direction w.r.t k_1) to v, k_1 is the (maximal) principal curvature, k_2 is the (minimal) principal curvature. We may assume $\lambda_1 = k_1$. Then

$$\lambda_2 = \cos^2\left(\frac{2\pi}{m}\right) k_1 + \sin^2\left(\frac{2\pi}{m}\right) k_2$$

$$\lambda_3 = \cos^2\left(2 \cdot \frac{2\pi}{m}\right) k_1 + \sin^2\left(2 \cdot \frac{2\pi}{m}\right) k_2$$

$$\vdots$$

$$\lambda_m = \cos^2\left((m-1)\frac{2\pi}{m}\right) k_1 + \sin^2\left((m-1)\frac{2\pi}{m}\right) k_2$$

Then

$$\sum_{i=1}^{m} \lambda_i = k_1\left(1 + \cos^2\left(\frac{2\pi}{m}\right) + \cdots \cos^2\left((m-1)\frac{2\pi}{m}\right)\right)$$

$$+ k_2\left(\sin^2\left(\frac{2\pi}{m}\right) + \cdots \sin^2\left((m-1)\frac{2\pi}{m}\right)\right)$$

Claim : $1+\cos^2\left(\frac{2\pi}{m}\right) + \cdots + \cos^2\left(\frac{m-1}{m}2\pi\right) = \frac{m}{2}$. $(m>2)$

If we can prove the claim, then

$$\sin^2\left(\frac{2\pi}{m}\right) + \cdots + \sin^2\left(\frac{m-1}{m}2\pi\right) = (m-1) - \left(\frac{m}{2}-1\right) = \frac{m}{2}.$$

$$\Rightarrow \sum_{i=1}^{m} \lambda_i = \frac{m}{2}(K_1+K_2) = mH.$$

Let's prove the claim. $\beta = \frac{2\pi}{m}$

$$\boxed{\cos^2 x = \frac{\cos 2x + 1}{2}}$$

double angle formula

$$\text{LHS} = 1+\cos^2\beta + \cos^2 2\beta + \cdots + \cos^2((m-1)\beta)$$

$$= 1 + \frac{(m-1)}{2} + \frac{1}{2}\left(\cos 2\beta + \cos 4\beta + \cdots + \cos(2(m-1)\beta)\right)$$

$$= 1 + \frac{m-1}{2} + \frac{1}{2}\cdot\frac{1}{2}\left(\sum_{k=-(m-1)}^{m-1} e^{i2k\beta} - 1\right) \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$= \frac{m}{2} + \frac{1}{4} + \frac{1}{4}\sum_{k=-(m-1)}^{m-1} e^{i2k\beta}$$

$$\sum_{k=-(m-1)}^{m-1} e^{i2k\beta} = \frac{e^{-i2(m-1)\beta}\left(1-e^{i2\beta(2m-1)}\right)}{1-e^{i2\beta}}, \quad (m>2)$$

$2m-1-m+1$

$$= \frac{e^{i2m\beta} - e^{-(m-1)i2\beta}}{e^{i2\beta}-1}$$

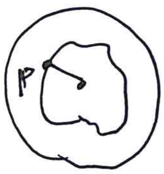
$$= \frac{e^{i(2m-1)\beta} - e^{-i(2m-1)\beta}}{e^{i\beta}-e^{-i\beta}}$$

$$= \frac{\sin(2m-1)\beta}{\sin\beta} \qquad (m>2) \qquad \beta = \frac{2\pi}{m}$$

$$= \frac{\sin(2\cdot 2\pi - \beta)}{\sin\beta} = -1.$$

$$\Rightarrow \quad LHS = \frac{m}{2}. \qquad \#$$

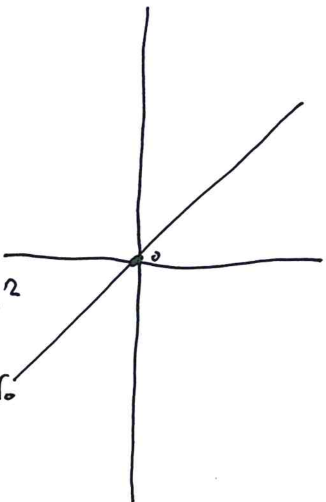
(3) Thm: On any cpt regular surface M in $\mathbb{R}^3$, $\exists$ a pt $p_0 \in M$ s.t $K(p_0) > 0$.

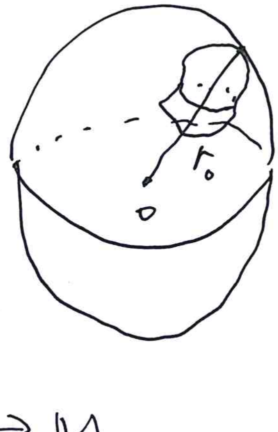


Pf: M is cpt $\Rightarrow$ $\exists\, p_0 \in M$ s.t

$$|p_0|^2 = \max_{p\in M} |p|^2 = r_0^2$$

i.e $\exists$ a sphere of radius r_0, $S^2_{r_0}$ s.t M is contained in $B(r_0)$ and touch $S^2_{r_0}$ at p_0.





Let $v \in T_{p_0}M$ and $|v| = 1$. $\alpha = $ a curve on M s.t $(-\varepsilon, \varepsilon) \to M$

$\alpha(0) = p_0$ and $\alpha'(0) = v$.

Then $|\alpha(t)|^2 \leq |p_0|^2 = |\alpha(0)|^2$

$$\Rightarrow \begin{cases} \left.\frac{d}{dt}\right|_{t=0} |\alpha(t)|^2 = 0 \\ \left.\frac{d^2}{dt^2}\right|_{t=0} |\alpha(t)|^2 \leq 0. \end{cases}$$

$$\Rightarrow \begin{cases} \langle \alpha(0), \alpha'(0) \rangle = 0 \\ \langle \alpha(0), \alpha''(0) \rangle + \langle \alpha'(0), \alpha'(0) \rangle \leq 0 \end{cases}$$

i.e $\begin{cases} \langle p_0, u \rangle = 0 \\ \langle p_0, \alpha''(0) \rangle \leq -|u|^2 = -1. \end{cases}$

Now $K_n(U) = \langle S_{p_0}(U), U \rangle = \langle S_{p_0}(\alpha'(0)), \alpha'(0) \rangle$

$= \langle \alpha''(0), \underset{\text{normal}}{U(p_0)} \rangle$

Note that at p_0, M is tangent to $S^2_{r_0}$ $\Rightarrow T_{p_0}M \,//\, T_{p_0}S^2_{r_0}$

So $U(p_0) = \dfrac{p_0}{|p_0|} = \dfrac{p_0}{r_0}$ (may $= -\dfrac{p_0}{r_0}$, we only discuss "+" case; "−" case is similar.)

$\Rightarrow \quad K_n(U) = \langle \alpha''(0), p_0 \rangle \dfrac{1}{r_0} \leq -\dfrac{1}{r_0} \quad \forall u \in T_{p_0}M$ with $|u|$=

$\Rightarrow \quad k_1, k_2 \leq -\dfrac{1}{r_0}$

$\Rightarrow \quad k_1 \cdot k_2 \geq \dfrac{1}{r_0^2} > 0$ at p_0.

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Ex: Let p be a pt on a regular surface M, k_i $(i=1,2)$ be the principal curvature at p and $\{u_i\}_{i=1,2}$ be a corresponding o.n. basis of principal direction. Let $k(\theta)$ be the normal curv at p in the direction of $\cos\theta\, u_1 + \sin\theta\, u_2$. Then

$$H = \frac{1}{2\pi}\int_0^{2\pi} k(\theta)\, d\theta .$$