Tutorial 7 17/10/2014

Let's recall the def. of normal curvature (different from Prof. Tam

$\forall$ unit vector $U \in T_pM$, the normal curvature of M in the U-direction is

$$K_n(U) = \langle S_p(U), U \rangle.$$

This def. is equ. to the def. of Prof. Tam.

α = a curve in M, T = the unit tangent of α ($\alpha'(s) = T$, s = arcl

U = Gauss map $\{T, U \times T, U\}$

$$\alpha'' = AT + BU \times T + CU$$

Since $\langle \alpha', \alpha' \rangle \equiv 1$,

$\Rightarrow \langle \alpha'', \alpha' \rangle = 0 \Rightarrow A = 0$

$B = K_g$ = geodesic curvature of α

$C = K_n$ = normal curvature of α

$$C = \langle \alpha'', U \rangle = \langle S(\alpha'), \alpha' \rangle = K_n(\alpha').$$

Geometric meaning of normal curvature :

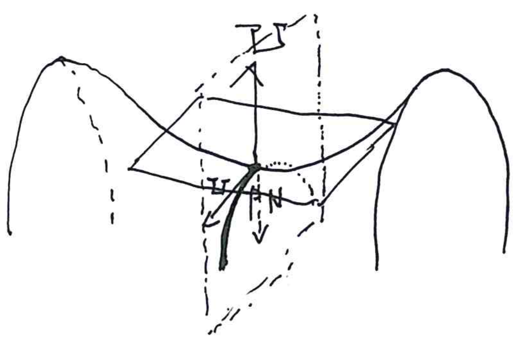
Prop (Prop 2.4.3 of Oprea) :

Let P = plane spanned by $U(p)$ and $\mathbf{u}$, where $\mathbf{u} \in T_pM$ and

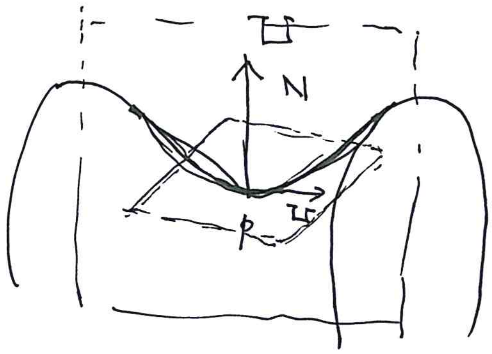
Let σ = unit speed curve formed by $P \cap M$ with $\sigma(0) = p$.

Then $k_n(\mathbf{u}) = \pm \kappa_\sigma(0)$ (κ_σ = curvature of σ).

(Note : the sign of $k(\mathbf{u})$ dep. on the choice of the surface normal U, i.e. the orientation of the surface.)



$k_n(\mathbf{u}) = -\kappa_\sigma(0)$



$k_n(\mathbf{u}) = +\kappa_\sigma(0)$

($\langle \alpha'', U \rangle = \langle S(\alpha'), \alpha' \rangle$, $\alpha'' = \kappa_\alpha N$ $(|\alpha'| = 1)$

$\Rightarrow$ $\kappa_\alpha \langle N, U \rangle = k_n$

$\Rightarrow$ $\kappa_\alpha \cos\theta = k_n$ $\quad\theta$ = angle between U and N.)

Pf: $\sigma'(0) \in P$, $\sigma'(0) \in T_pM$

$\Rightarrow$ $\sigma'(0) = \pm \mathbf{u}$

(If necessary, we reverse the parametrization of σ.

We have $\sigma'(0) = v$

$\sigma \in P$ is a plane curve

$N \in P$, $N \perp \sigma'(0) = v$

$\Rightarrow \quad N = \pm U$

$\Rightarrow \quad k_n(v) = \cos\theta \, k_\sigma(0) = \pm k_\sigma(0)$.

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$k_n(v) = \langle S_p(v), v \rangle \qquad k_n : S^1 \to \mathbb{R}$, $\quad S^1 \subset T_pM$, $\quad S^1$ cpt

$\exists \; u_1, u_2 \in S^1$ s.t

$$\begin{cases} k_n(u_1) = k_1 = \max\limits_{v \in S^1} k(v) \\ k_n(u_2) = k_2 = \max\limits_{v \in S^1} k(v) . \end{cases}$$

Euler formula :

Let v_1 and v_2 be 2 orthonormal eigenvectors of S_p s.t

$$\lambda_1 = k_n(u_1) \qquad \begin{pmatrix} S_p(u_1) = \lambda_1 u_1 \\ S_p(u_2) = \lambda_2 u_2 \end{pmatrix}$$

$$(\geq)$$

$$\lambda_2 = k_n(u_2)$$

then for $u = \cos\theta \, u_1 + \sin\theta \, u_2 \in S^1 \subset T_pM$, we have

$$k_n(u) = \cos^2\theta \, \lambda_1 + \sin^2\theta \, \lambda_2$$

Pf: $k(v) = \langle S_p(v), v \rangle = \langle S_p(\cos\theta \, u_1 + \sin\theta \, u_2), \cos\theta \, u_1 + \sin\theta \, u_2 \rangle$

$$= \cos^2\theta \, k_n(u_1) + 2\sin\theta\cos\theta \underbrace{\langle \underbrace{S_p(u_1)}_{\lambda_1 u_1}, u_2 \rangle}_{0} + \sin^2\theta \, k_n(u_2)$$

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So $\lambda_1 = \max$, $\lambda_2 = \min$.

$\{ K(u) = (1-t)\lambda_1 + t\lambda_2,\ 0 \le t \le 1 \}$, $\cancel{\lambda_2 + \cos^2\theta(\lambda_1 - \lambda_2)}$

Def :

Gauss curvature

$$K(p) = \det(S_p)$$

Mean curvature (Check well-def)

$$H(p) = \frac{1}{2}\operatorname{trace}(S_p)$$

One can choose a basis of T_pM s.t

S_p is represented by $\begin{bmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{bmatrix}$

So $\begin{cases} K(p) = k_1(p)\cdot k_2(p) \\ H(p) = \frac{1}{2}(k_1(p) + k_2(p)) \end{cases}$

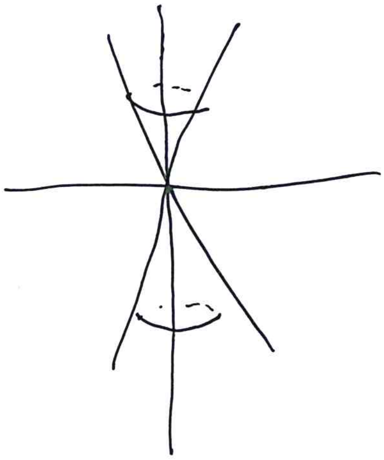
Ex 1: $X(u,v) = (v\cos u, v\sin u, v)$. Compute the Gauss map and its differential. Estimate the area of the image of Gauss map on the sphere.

Gauss map : $X_u = (-v\sin u, v\cos u, 0)$, $X_v = (\cos u, \sin u, 1)$

$$X_u \times X_v = (v\cos u, v\sin u, -v)$$

$$N = \frac{X_u \times X_v}{\cdots} = \frac{v}{\cdots}(\cos u, \sin u, -1) = \frac{1}{\sqrt{\cdots}}(\cos u, \sin u, -1)$$

We consider $u \in [0, 2\pi)$, $v \in (0, +\infty)$, $v=0$ is a singularity



$(g_*)_p = -S_p$ if we consider $T_{g(p)}S^2 = T_pM$ in the natural way

We can compute the matrix representation of S_p under $[X_u, X_v]$.

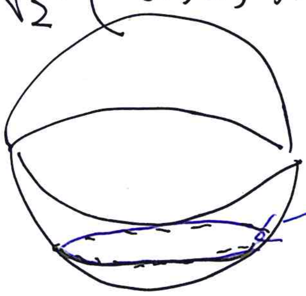
$E = v^2$, $F = 0$, $G = 2$, $l = -\frac{1}{\sqrt{2}}v$, $m = 0$, $n = 0$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}v} & 0 \\ 0 & 0 \end{pmatrix} \qquad \frac{1}{EG-F^2}\begin{pmatrix} Gl - Fm & Gm - Fn \\ -Fl + Em & -Fm + En \end{pmatrix}$$

So we may write the matrix representation of $(g_*)_p$ under $[X_u, X_v]$

by $\begin{pmatrix} \frac{1}{\sqrt{2}v} & 0 \\ 0 & 0 \end{pmatrix}$.

$U = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$ is just a circle



So Area $(g(X)) = 0$.

Ex2 : Show that the Gauss curvature of a ruled surface

$X(u,v) = \beta(u) + v\delta(u)$ is

$$K = - \frac{\langle \beta', \delta \times \delta' \rangle^2}{|\beta' \times \delta + v\delta' \times \delta|^4} \triangleq (*).$$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}}$$

$X_u = \beta' + v\delta'$, $X_v = \delta$, $X_{uu} = \beta'' + v\delta''$

$X_{uv} = \delta'$, $X_{vv} = 0$

$X_u \times X_v = \beta' \times \delta + v\delta' \times \delta$, $\quad U = \dfrac{\beta' \times \delta + v\delta' \times \delta}{|\beta' \times \delta + v\delta' \times \delta|}$

$$h_{11} = \langle X_{uu}, U \rangle = \frac{\langle \beta' \times \delta, \beta'' \rangle + v^2 \langle \delta' \times \delta, \delta'' \rangle + \cdots}{|\beta' \times \delta + v\delta' \times \delta|}$$

$h_{22} = 0$

$$h_{12} = \langle X_{uv}, U \rangle = \frac{\langle \beta' \times \delta, \delta' \rangle}{|\beta' \times \delta + v\delta' \times \delta|}$$ parallelepi

$\det \mathrm{I} = |X_u \times X_v|^2 \quad \left(|a \times b|^2 = |a|^2 |b|^2 - \langle a, b \rangle^2 \right)$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{1}{|X_u \times X_v|^2} \cdot \frac{-\langle \beta' \times \delta, \delta' \rangle^2}{|\beta' \times \delta + v\delta' \times \delta|^2} = (*).$$

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