Tutorial 13 28/11/2014

① Ex: M = regular surface

α = unit speed curve on M with length L

$\beta_\varepsilon(s) = \alpha(s) + \varepsilon\delta(s)$, $\varepsilon \in$ [small interval containing 0]

$\langle \delta, \alpha' \rangle = 0$ and $\langle \delta, \mathbb{U} \rangle = 0$

$\mathbb{U}$ = Gauss map (i.e unit normal)

$$\Longrightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\beta_\varepsilon) = -\int_0^L k_g \langle \mathbb{U} \times T, \delta \rangle \, ds.$$

Rmk: If α = a geodesic, then α is a critical pt of such length variation.

Pf: $\beta_\varepsilon' = \alpha' + \varepsilon\delta'$, $|\beta_\varepsilon'| = \sqrt{1+\varepsilon^2|\delta'|^2 + 2\varepsilon\langle \alpha', \delta' \rangle}$.

$$L(\beta_\varepsilon) = \int_0^L \sqrt{1+\varepsilon^2|\delta'|^2 + 2\varepsilon\langle \alpha', \delta' \rangle} \, ds$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\beta_\varepsilon) = \int_0^L \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \sqrt{1+\varepsilon^2|\delta'|^2 + 2\varepsilon\langle \alpha', \delta' \rangle} \, ds$$

$$= \int_0^L \frac{1}{2} \cdot \left(2\varepsilon|\delta'|^2 + 2\langle \alpha', \delta' \rangle\right)\Big|_{\varepsilon=0} ds$$

$$= \int_0^L \langle \alpha', \delta' \rangle \, ds$$

$$\langle \delta, \alpha' \rangle = 0 \Rightarrow \quad \langle \delta', \alpha' \rangle + \langle \delta, \alpha'' \rangle = 0.$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\beta_\varepsilon) = -\int_0^L \langle \delta, \alpha'' \rangle \, ds$$

$$= -\int_0^L \langle \delta, (\alpha'')_{tan} \rangle \, ds \quad \text{Since } \langle \delta, U \rangle = 0$$

$$= -\int_0^L k_g \langle \delta, U \times T \rangle \, ds.$$

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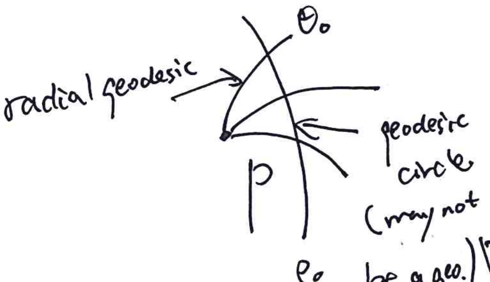
② Geodesic Polar Coordinates

(At any pt in a regular surface, we can find a local coordinate around that pt s.t. $g = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}$.)

Let $p \in M$, $\{e_1, e_2\}$ be an orthonormal basis of T_pM. For any unit vect. $w = \cos v\, e_1 + \sin v\, e_2 \in T_pM$, we can find $\alpha_{p,w}(u)$, a unit speed geodesic with $\begin{cases} \alpha_{p,w}(0) = p \\ \alpha'_{p,w}(0) = w \end{cases}$ in an interval $(-\varepsilon_{p,w}, \varepsilon_{p,w})$.



Let v runs over $[0, 2\pi)$, then we get a coordinate $X(u,v) = \alpha_{p,\, \cos v e_1 + \sin v e_2}(u)$.

But we should check $\varepsilon_{p,\, \cos v e_1 + \sin v e_2}$ have a uniform lower bdd (of v). See §4.6 of do Carmo for rigorous discussion.

$g_{11} = \langle X_u, X_u \rangle = |\alpha'|^2 = 1.$ $\qquad (u, v) \triangleq (\rho, \theta) \, (= (r, \theta$

$\alpha_{p, \cos\theta e_1 + \sin\theta e_2}(u)$ is a geodesic $\Rightarrow$

$(\rho = t, \ \theta = \text{Constant})$

$$\begin{cases} \dfrac{d^2\rho}{dt^2} + \Gamma^1_{11}\left(\dfrac{d\rho}{dt}\right)^2 + 2\Gamma^1_{12}\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \Gamma^1_{22}\left(\dfrac{d\theta}{dt}\right)^2 = 0 \\ \dfrac{d^2\theta}{dt^2} + \Gamma^2_{11}\left(\dfrac{d\rho}{dt}\right)^2 + 2\Gamma^2_{12}\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \Gamma^2_{22}\left(\dfrac{d\theta}{dt}\right)^2 = 0 \end{cases}$$

$$\Rightarrow \quad \Gamma^1_{11} = \Gamma^2_{11} = 0$$

$$\Rightarrow \quad \begin{cases} g^{1l}(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}) = 0 \\ g^{2l}(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}) = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} g^{12}(2\partial_1 g_{12}) = 0 \\ g^{22}(2\partial_1 g_{12}) = 0 \end{cases}$$

$\Rightarrow \quad \partial_1 g_{12} = 0$ (*) Since g^{12}, g^{22} can not vanish at the same time

~~And note that~~ $\cancel{g_{12} = \langle X_1, X_2 \rangle = 0}$ ~~at~~ $\cancel{p}$ ~~since~~ $\cancel{X_2(p) = \frac{d}{d\theta}(\alpha(0)_{p,\cos\theta e_1 + \sin\theta e_2})}$

And

$\cancel{\Rightarrow g_{12} = 0.}$ $\quad |g_{12}| \leq |\langle X_1, X_2 \rangle| = |X_2| \overset{(**)}{\cancel{= 0}}.$

$\to 0$ as $\rho \to 0$

~~We want to show more :~~ Since $\lim_{\rho \to 0} \frac{\partial}{\partial\theta} \alpha_\theta(\rho) = 0$ (Check it!

~~$g_{22} = 0$~~
~~$\rho \to 0$~~

By ⊛⊛, we get $g_{12} \equiv 0$.

We would like to change coordinate to see more.

$$\begin{cases} u = \rho\cos\theta \\ v = \rho\sin\theta \end{cases} \qquad \left((u,v) \triangleq \text{normal coordinate} \right)$$

$\theta_0 =$ Constant (radial geodesic) $\Leftrightarrow$ $(u,v) = t(u_0, v_0)$

In particular,

$$\underset{\text{polar coordinate}}{X(t,0)} = \alpha_{p,e_1}(t) \qquad ⊛$$

$$X_u(0,0) = \alpha'_{p,e_1}(0) = e_1 \quad \left(\underset{\text{normal}}{X_u(0,0) = e_1} \right)$$

$$X(t, \tfrac{\pi}{2}) = \alpha_{p,e_2}(t)$$

$$X_v(0, \tfrac{\pi}{2}) = \alpha'_{p,e_2}(0) = e_2 \quad \left(\underset{\text{normal}}{X_v(0,0) = e_2} \right)$$

normal

$$\Rightarrow \quad g(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$(u^1, u^2) \triangleq (u,v) = t(u_0, v_0)$

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0, \quad k = 1,2$$

$$\Rightarrow \quad \Gamma^k_{ij}\, u_0^i \cdot u_0^j = 0 \qquad (u_0^1, u_0^2) \triangleq (u_0, v_0)$$

Since (u_0, v_0) is arbitrary as long as $u_0^2 + v_0^2 = 1$

$\Rightarrow \quad \Gamma^k_{ij} = 0$ along the radial geodesic $\forall i,j,k = 1,2$

In particular, $\Gamma^k_{ij}(p) = 0$, $\forall\, i,j,k = 1,2$.

$$\boxed{X_{ij} = b_{ij}\vec{n} + \Gamma_{ij}^{k} X_k}$$

$$\frac{\partial}{\partial u^k} g_{u^i u^j} = \frac{\partial}{\partial u^k} \langle X_{u^i}, X_{u^j} \rangle$$

$$= \langle X_{u^i u^k}, X_{u^j} \rangle + \langle X_{u^i}, X_{u^j u^k} \rangle$$

$$= \langle \Gamma_{ik}^{s} X_{u^s}, X_{u^j} \rangle + \langle X_{u^i}, \Gamma_{jk}^{s} X_{u^s} \rangle$$

$$= 0 \text{ at } p.$$

$\Rightarrow \quad g_{u^i u^j} = \delta_{ij} + 2^{nd}$ or higher order terms.

$$\sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & g_{22}(\rho,\theta) \end{pmatrix}} = \sqrt{\det(g_{u^i u^j})}\, \frac{\partial(u,v)}{\partial(\rho,\theta)} \leftarrow (\text{ Exercise! })$$

where $\dfrac{\partial(u,v)}{\partial(\rho,\theta)} = \det\begin{pmatrix} \frac{\partial u}{\partial \rho} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial \rho} & \frac{\partial v}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} = \rho$

$$\Rightarrow \quad \sqrt{G(\rho,\theta)} = \rho\sqrt{\det(g_{u^i u^j})}$$

$$\Rightarrow \quad \lim_{\rho\to 0} \sqrt{G} = 0$$

$$\left(\sqrt{G(\rho,\theta)}\right)_\rho = \sqrt{\det(g_{u^i u^j})} + \rho\left(\frac{\partial}{\partial \rho}\sqrt{\det(g_{u^i u^j})}\right)$$

$$\to 1 \quad \text{as } \rho \to 0.$$

$$\cancel{|X_{12}| = |\langle X_1, X_2 \rangle| \leq |X_1|\,|X_2| = |X_2| \to 0 \text{ as } \rho \to 0}$$

~~Together with the fact that~~

$\cancel{\partial_1 g_{12} = 0}$ ~~along the radial geodesic~~

$\cancel{g_{12} = 0}$

Claim: (i) $\sqrt{G}\,(\rho,\theta) = \rho - \frac{\rho^3}{3!}K(p) + R(\rho,\theta)$

where $\lim_{\rho\to 0} \frac{R(\rho,\theta)}{\rho^3} = 0$.

(ii) $K(p) = \lim_{\rho_0 \to 0} \frac{3}{\pi} \frac{2\pi\rho_0 - L_0}{\rho_0^3}$ where L_0 = length of the geodesic circle $\rho \equiv \rho_0$

To see (i), we need to show

$$\lim_{\rho\to 0}\sqrt{G} = 0, \quad \lim_{\rho \to 0} (\sqrt{G})_\rho = 1, \quad \lim_{\rho\to 0} (\sqrt{G})_{\rho\rho} = 0$$

and $\lim_{\rho \to 0} (\sqrt{G})_{\rho\rho\rho} = -K(p)$. (Although there is no def. at p, we may use limit to def. it.)

Recall a formula for Gauss curvature in the case of $F=0$:

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$$= -\frac{1}{2\sqrt{G}}\left(\frac{\partial}{\partial\rho}\left(\frac{G_\rho}{\sqrt{G}}\right)\right) \qquad \left(\frac{G_\rho}{\sqrt{G}} = 2\frac{\partial}{\partial\rho}(\sqrt{G})\right)$$

$$= -\frac{1}{2\sqrt{G}}\, 2(\sqrt{G})_{\rho\rho}$$

□

$$(\sqrt{G})_{\rho\rho} = -\sqrt{G}\,K$$

$$\lim_{\rho\to 0} (\sqrt{G})_{\rho\rho} = 0$$

$$(\sqrt{G})_{\rho\rho\rho} = -(\sqrt{G})_{\rho} K - K_{\rho}\cdot\sqrt{G}$$

$$\lim_{\rho\to 0} \quad \downarrow \quad = -K(p).$$

(ii) $$L_0 = \int_0^{2\pi} \sqrt{G}(\rho_0,\theta)\,d\theta$$

$$= \int_0^{2\pi} \left(\rho_0 - \frac{\rho_0^3}{3!}K(p) + R(\rho_0,\theta)\right) d\theta$$

$$= 2\pi\rho_0 - \frac{\pi}{3}\rho_0^3 K(p) + \int_0^{2\pi} R(\rho_0,\theta)\,d\theta$$

$$\Rightarrow K(p) = \frac{3}{\pi}\left[\left(\frac{2\pi\rho_0 - L_0}{\rho_0^3}\right)\right] + \frac{3}{\pi}\int_0^{2\pi} \frac{R(\rho_0,\theta)}{\rho_0^3}\,d\theta$$

$$= \lim_{\rho_0\to 0} \frac{3}{\pi}\cdot\frac{2\pi\rho_0 - L_0}{\rho_0^3}$$

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