Tutorial 12 21/11/2014

① 2 Exercises in last tutorial.

(a) M = cpt regular surface in $\mathbb{R}^3$

g = a C^∞ function on M s.t $\forall f$ C^∞ function on M

$$\int_M g \cdot f \, dA = 0$$

$$\Rightarrow \quad g \equiv \text{Constant}.$$

Pf:
$$C = \frac{\int_M g \, dA}{\int_M dA}, \quad \bar{g} = g - c$$

Then $\int_M \bar{g} \, dA = 0$.

$$\int_M \bar{g}^2 \, dA = \underbrace{\int_M g \cdot \bar{g} \, dA}_{\text{assumption}} - c \int_M \bar{g} \, dA$$

$$= 0 - 0$$

$$= 0.$$

$$\Rightarrow \bar{g}^2 = 0 \Rightarrow g \equiv c.$$

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(b) D = domain in $\mathbb{R}^2$

f = cts function on D s.t $\forall g$ = C^∞ function with

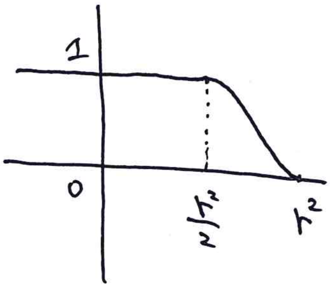
$g|_{\partial D} = 0$, $\int_D f \cdot g \, dx dy = 0$

$\Rightarrow \quad f \equiv 0.$

Pf: We argue by contradiction. Suppose $\exists\, p \in D$ s.t. $f(p) > 0$, ("< 0" is similar) then $\exists$ a neighborhood of p, Ω s.t. $f > 0$ on Ω

$r \triangleq \mathrm{dist}(p, \partial\Omega)$.

$\eta = C^\infty$ cut-off function



(Please explicitly find such η. Hint: $e^{-\frac{1}{x}}$)

$g(x) \triangleq \eta(\mathrm{dist}^2(x,p))$ $\quad C^\infty$ function on D with $g|_{\partial D} = 0$

$$\int_D f \cdot g \, dx dy = \int_{B(p,r)} f \cdot g \, dx dy$$

> 0 Since in this ball, $f, g > 0$

$\rightarrow\leftarrow$ with the assumption.

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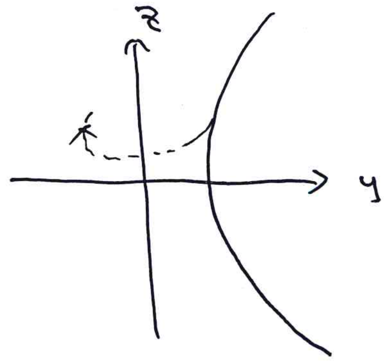
② Alexandrov's Thm (1956) If M is a cpt embedded (i.e. no self-intersection) surface of constant mean curvature, then M is a standard sphere.

Examples:

Minimal surface: $\mathbb{R}^2$, catenoid $X(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$

Constant mean curvature surface ($H = \text{Constant} > 0$)

$\mathbb{S}^2$.



To prove Alexandrov's Thm, we need

Ros' Thm: M = cpt embedded surface in $\mathbb{R}^3$ bounding a domain D of volume V. If $H > 0$ on M, then

$$\iint_M \frac{1}{H} \, dA \geq 3V.$$

"=" holds $\Leftrightarrow$ M is a standard sphere.

Pf of Alexandrov's Thm:

M cpt $\Rightarrow$ $\exists\, p_0 \in M$ s.t $K(p_0) > 0$.

i.e $k_1(p_0)$, $k_2(p_0)$ are of same sign. One can just choose inward normal makes $k_1(p_0), k_2(p_0) > 0$

$\Rightarrow H(p_0) > 0$

$\Rightarrow H > 0$.

Now we need to apply 1st Variational formular for area:

$$A'(0) = -\iint_M 2H \langle U, V \rangle \, dA$$

$M : X(u,v)$

$M^t : \quad X^t(u,v) = X(u,v) + tV(u,v)$ for $|t|$ small

$A(t) = \text{Area}(M^t)$, $A(0) \triangleq A$.

Take a special case $V = X$.

Then $\quad X^t = (1+t)X$

$$A(t) = (1+t)^2 A$$

$$2A = A'(0) = -\iint_M 2H \langle U, X \rangle \, dA$$

$$\Rightarrow A = -\iint_M H \langle U, X \rangle \, dA.$$

$$= -H \iint_M \langle X, U \rangle \, dA$$

inward

$\left(\begin{array}{l} \text{div. Thm:} \\ \iiint_D \operatorname{div} \vec{F} \, dvol = \iint_{\partial D} \langle \vec{F}, \vec{n} \rangle \\ \vec{n} = \text{outward normal} \end{array} \right.$

By div. Thm

$$= H \iiint_D \operatorname{div}(X) \, dvol$$

$$= H \iiint_D \left(\frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} \right) dvol$$

$$= 3HV$$

By Ros' Thm,

$$\iint_M \frac{1}{H}\, dA \geq 3V \text{ and " = " holds iff } M \text{ is a standard sphere}$$

$\quad\ \|$ now

$$\frac{1}{H}A = 3V.$$

So M is a standard sphere. #

Proof of Ros' Thm:

D = domain bdd by M

Locally, D can be represented by

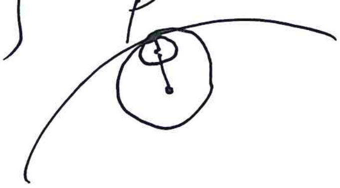
$$Y(u,v,t) = X(u,v) + t\,U(u,v)$$

where X = local parametrization of M

U = inward normal (unit)

$t \in [0, h(u,v)]$

with $h(p) = h(u,v) = \sup\{r \mid$ the pt p is the unique nearest pt on M to the pt q at a distance r from p along the normal $U\}$



Vol of D_ε in terms of $\{Y(u,v,t)\}$.

Note that if we choose $\{X(u,v)\}$ so that the coordinate patches have no interior intersection, then the ~~coor~~ correspond $\{Y(u,v,t)\}$ also have no interior intersection. And we can calculate Vol(D) by summing up the volumes of each local coordinates $Y(u,v,t)$. (See P190 "Oprea's book)

Hence

$$\mathrm{Vol}(D) = \iint \left[\int_0^{h(u,v)} |\langle Y_u \times Y_v, Y_t\rangle| \, dt\right] du\, dv$$

$$Y = X + tU, \quad \begin{cases} Y_u = X_u + tU_u \\ Y_v = X_v + tU_v \\ Y_t = U \end{cases}$$

One can choose a coordinate patch s.t

$$\begin{cases} U_u = -S(X_u) = -k_1 X_u \\ U_v = -S(X_v) = -k_2 X_v \end{cases}$$

(What if we can't choose this coordinate patch?

Fact: neighorhood of a umbilical pt, we can do this.)

$$\Rightarrow \begin{cases} Y_u = (1-k_1 t) X_u \\ Y_v = (1-k_2 t) X_v \end{cases}$$

$$Y_u \times Y_v = (1-k_1 t)(1-k_2 t)\, X_u \times X_v$$

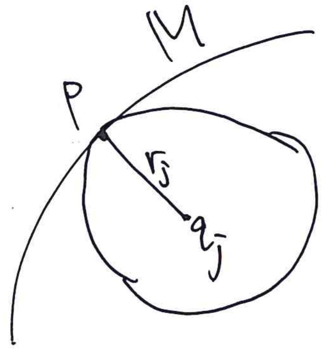
$$= (1-2Ht + Kt^2)\, X_u \times X_v$$

$$\Rightarrow |\langle Y_u \times Y_v, Y_t\rangle| = \underbrace{|1-2Ht+Kt^2|}_{\downarrow} \cdot |\langle X_u \times X_v, \nu\rangle|$$

$$\Rightarrow \qquad = \qquad\qquad |X_u \times X_v|.$$

$$\mathrm{Vol}(D) = \iint_M \left(\int_0^{h(u,v)} |1-2Ht+Kt^2|\, dt \right) dA$$

Claim: $\frac{1}{h(p)} \geq \max\{k_1(p), k_2(p)\}, \ \forall p \in M.$



$r_j \to h(p)$

Using the same method in the proof of → cpt M, $\exists p$ s.t $K(p)$:

$\Rightarrow \quad \max\{k_1(p), k_2(p)\} \leq \frac{1}{r_j}$

Let $j \to +\infty$, we get the claim.

$$1-2Ht+Kt^2 = (1-k_1 t)(1-k_2 t) \geq \left(1-\frac{1}{h}\cdot t\right)^2 \geq 0.$$

$$\left(\begin{array}{l} 1-k_1 t \overset{?}{\geq} 0 \\ \quad k_1 \overset{?}{\leq} \frac{1}{t} \text{ Since } \frac{1}{h} \leq \frac{1}{t}. \end{array} \right) \qquad \left(k_1 \overset{?}{\leq} \frac{1}{h}.\ \text{v.} \right)$$

$$\int_0^{h(p)} |\cdots|\, dt = \int_0^{h(p)} (1-2Ht+Kt^2)\, dt$$

$$= \int_0^{h(p)} \left[(1-Ht)^2 - \overset{\geq 0}{(H^2-K)}\, t^2 \right] dt$$

$$\int_0^{h(p)} |\cdots| \leq \int_0^{h(p)} (1-Ht)^2\,dt$$

$$\leq \int_0^{\frac{1}{H}} (1-Ht)^2\,dt \qquad \left(H = \frac{k_1+k_2}{2} \leq \frac{1}{h}\right)$$

$$= \frac{1}{3H}.$$

$$\Rightarrow \quad Vol(D) = \iint_M \left(\int_0^{h(p)} |\cdots|\,dt \right) dA$$

$$\leq \frac{1}{3} \iint_M \frac{1}{H}\,dA.$$

Finally, "=" holds $\Rightarrow$ all the ineq. have to be "=".

In particular, $H^2 - K = 0$

$\Rightarrow$ $k_1 = k_2$ $\Rightarrow$ totally umbilical

$\xRightarrow{(H>0)}$ Standard sphere.

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