Tutorial 11 14/11/2014

① Does there exist a surface $X(u,v)$ with

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix} \text{ and } (h_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix} ?$$

This surface doesn't exist! Because it can not satisfy the Codazzi equation. Recall the Codazzi equation:

$$(*) \quad \frac{\partial h_{ij}}{\partial u^k} + \sum_l \Gamma_{ij}^l h_{lk} = \frac{\partial h_{ik}}{\partial u^j} + \sum_l \Gamma_{ik}^l h_{lj}, \quad \forall i,j,k=1,2.$$

We can compute the Christoffel symbols of (g_{ij}):

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = 0$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = -\tan u, \quad \Gamma_{22}^1 = \sin u \cos u$$

Take $i=j=2$, $k=1$, then

$$\text{LHS of } (*) = 0 + \sin u \cos u \cdot \cos^2 u = \sin u \cdot \cos^3 u$$

$$\text{RHS of } (*) = 0 + (-\tan u) \cdot 1 = -\tan u$$

But this is impossible!

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(2) Show that if M is cpt, connected, oriented constant mean curvature surface with positive Gauss curvature, then M is a standard sphere.

Pf: Consider $f(x) = H^2 - K = \dfrac{(k_1(x) - k_2(x))^2}{4}$.

Since M is cpt, we assume $p = \{x \in M \mid \max\limits_{x \in M} f(x)\}$.

If $f(p) = 0$, then M is consisting entirely of umbilical pts.

Then we see that M has constant Gauss curvature K (positive)

$\Rightarrow$ M is a standard sphere with radius $\frac{1}{\sqrt{K}}$.

If $f(p) > 0$, note that $p = \{x \in M \mid \max\limits_{x \in M} \underbrace{(k_1 - k_2)(x)}_{\triangleq\, g(x)}\}$

(At all pts, we assume $k_1 \geq k_2$). By assumption, $k_1 + k_2 \equiv C$ where C is some constant.

Note that $g(x) + C = 2k_1(x)$, so if $g(x)$ takes maximum, $k_1(x)$ also takes maximum and $k_2(x)$ takes minimum.

i.e $\quad k_1(p) = \max\limits_M k_1 > k_2(p) = \min\limits_M k_2$

By Hilbert's lemma, we have $K(p) \leq 0$ $\rightarrow\leftarrow$.

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(3) Area minimization with a fixed boundary $\Rightarrow H \equiv 0$

M = a Monge patch $X(u,v) = (u, v, f(u,v))$ of least area with boundary C.

D = domain of f with boundary curve ∂D s.t $f(\partial D) = C$.

Consider the family of nearby surfaces of the form

$$M^t : \ z^t(x,y) = f(x,y) + t\,g(x,y) \ , \quad |t| \text{ small}$$

where g is a function def. on D s.t $g|_{\partial D} = 0$.

A Monge patch of M^t is given by

$$X^t(u,v) = (u, v, f(u,v) + t\,g(u,v)).$$

Then
$$\begin{cases} X_u^t = (1, 0, f_u + t g_u) \\ X_v^t = (0, 1, f_v + t g_v) \end{cases}$$

$$\Rightarrow \ |X_u^t \times X_v^t| = |(-(f_u + t g_u), -(f_v + t g_v), 1)|$$

$$= \sqrt{1 + (f_u + t g_u)^2 + (f_v + t g_v)^2}$$

$$= \sqrt{1 + f_u^2 + f_v^2 + 2t(f_u g_u + f_v g_v) + t^2(g_u^2 + g_v^2)}$$

$$= \sqrt{1 + |Df|^2 + 2t\langle Df, Dg\rangle + t^2 |Dg|^2}.$$

$$A(t) \triangleq A(M^t) = \iint_D \sqrt{1+|\nabla f|^2 + 2t\langle \nabla f, \nabla g\rangle + t^2|\nabla g|^2}\; du\,dv$$

$$\Rightarrow \frac{dA}{dt} = \iint_D \frac{\langle \nabla f, \nabla g\rangle + t|\nabla g|^2}{\sqrt{1+|\nabla f|^2+2t\langle \nabla f, \nabla g\rangle + t^2|\nabla g|^2}}\; du\,dv$$

Since M is area minimizing (w.r.t the fixed boundary C),

$\frac{dA}{dt}(0) = 0$. i.e

$$\iint_D \frac{\langle \nabla f, \nabla g\rangle}{\sqrt{1+|\nabla f|^2}}\; du\,dv = 0 \quad \text{or}$$

$$\iint_D \left[\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}}\right) g_u + \left(\frac{f_v}{\sqrt{1+|\nabla f|^2}}\right) g_v\right] du\,dv = 0$$

By Green's Thm $\oint_C P\,du + Q\,dv = \iint_D \left(\frac{\partial P}{\partial v} + \frac{\partial Q}{\partial u}\right) du\,dv$

$$\Longrightarrow \quad Q = \frac{f_u g}{\sqrt{1+|\nabla f|^2}}, \quad P = \frac{-f_v g}{\sqrt{1+|\nabla f|^2}}$$

$$0 = \iint_D \frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1+|\nabla f|^2}}\right) g + \frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}}\right) g \; du\,dv$$

i.e $$\iint_D \left[\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1+|\nabla f|^2}}\right) + \frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}}\right)\right] g \; du\,dv = 0.$$

Sine g is a arbitrary function with $g|_{\partial D} = 0$, we have

$$\frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1+\|\nabla f\|^2}}\right) + \frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1+\|\nabla f\|^2}}\right) = 0$$

(Ex: Show that if f = cts function on a domain $D \subseteq \mathbb{R}^2$ s.t

$$\iint_D f \cdot g = 0 \quad \forall g \ C^\infty \text{ in } D \text{ and } g|_{\partial D} = 0, \text{ then } f \equiv 0 \text{ on } D$$

$$\Rightarrow \frac{f_{uu}}{\sqrt{1+\|\nabla f\|^2}} - \frac{f_u(f_u f_{uu} + f_v f_{uv})}{(1+\|\nabla f\|^2)^{3/2}} + \frac{f_{vv}}{\sqrt{1+\|\nabla f\|^2}} - \frac{f_v(f_u f_{uv} + f_v f_{vv})}{(1+\|\nabla f\|^2)^{\frac{3}{2}}} =$$

$$\Rightarrow (1+f_v)^2 f_{uu} - 2 f_u f_v f_{uv} + (1+f_u)^2 f_{vv} = 0$$

i.e $H = 0$.

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Ex: Show that if the mean curvature H of a cpt regular surface M satisfies $\iint_M f H \, dA = 0 \ \forall f$ with $\iint_M f \, dA = 0$

Then H is a constant.

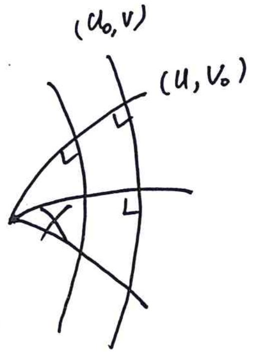
④ Geodesic is the shortest path joining with 2 fixed pts locally. (~~Global~~ "Global" is wrong!)

Locally, one can find a coordinate patch of $x \in M$ s.t

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} \quad \text{i.e. } |X_u|^2 = 1,\ \langle X_u, X_v \rangle = 0$$

$$\text{and } |X_v|^2 = G(u,v) > 0$$

We call it geodesic polar coordinate of x. (Ex: Existence of this coordinate.)



Then $\alpha(u) = X(u, v_0)$ for a fixed v_0 is a geodesic parametrized by arc-length

Pf: $\begin{cases} \alpha' = X_1 \\ \alpha'' = X_{11} \end{cases}$

$$\Rightarrow \quad |\alpha'|^2 = |X_1|^2 = 1 \quad \Rightarrow \quad u = \text{arc-length}$$

$$\langle \alpha'', X_1 \rangle = \langle X_{11}, X_1 \rangle = \tfrac{1}{2} \langle X_1, X_2 \rangle_1 = 0$$

$$\langle \alpha'', X_2 \rangle = \langle X_{11}, X_2 \rangle = \cancel{\langle X_1, X_2 \rangle_1}^{\,0} - \langle X_1, X_{21} \rangle$$

$$= -\tfrac{1}{2} \langle X_1, X_1 \rangle_2 = 0$$

$$\Rightarrow \quad (\alpha'')_{tan} = 0 \quad \Rightarrow \quad \alpha \text{ is a geodesic.} \qquad \#$$

Now we consider another curve $\beta(s)$, $s \in [0, L]$ where

L is the length of $\alpha(u)$ s.t

$$\begin{cases} \beta(0) = \alpha(0) = X(0, v_0) \\ \beta(L) = \alpha(L) = X(L, v_0) \end{cases}$$

Write $\beta(s) = X(u(s), v(s))$. So $u(0) = 0$, $u(L) = L$

$$\beta' = X_1 \cdot u' + X_2 \cdot v'$$

$$L(\beta^*) = \int_0^L \sqrt{\langle \beta', \beta' \rangle}\, ds$$

$$= \int_0^L \sqrt{u'^2 + G v'^2}\, ds \geq \int_0^L |u'|\, ds \geq \int_0^L u'\, ds$$

$$= u(L) - u(0) = L$$

$L(\beta) = L(\alpha) \Leftrightarrow u' > 0$ and $v' \equiv 0$

$$\Rightarrow \beta(s) = X(u(s), v_0)$$

i.e $\beta = \alpha$ up to a reparametrization.

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