Tutorial 10 07/11/2014

① An exercise in last tutorial.

Let $X(u,v) = (u, v, g(u) + h(v))$ be a minimal surface. Show that

$$(1+h'(v)^2)\, g''(u) + [1+g'(u)^2]\, h''(v) = 0.$$

Try to find explicit $f(u,v) = g(u) + h(v)$ for some domain $D \left(\subseteq \mathbb{R}^2 \right.$ $\ni (u,$

We consider a little general case (graph) $X(u,v) = (u, v, f(u,v))$

$$X_u = (1, 0, f_u) \qquad X_{uu} = (0,0,f_{uu}) \qquad X_u \times X_v = (-f_u, -f_v, 1)$$

$$X_v = (0, 1, f_v) \qquad X_{uv} = (0,0,f_{uv}) \qquad$$

$$X_{vv} = (0,0,f_{vv}) \qquad U = \frac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}}$$

$$E = 1+f_u^2, \quad F = f_u f_v, \quad G = 1+f_v^2.$$

$$l = \frac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}}, \quad m = \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}, \quad n = \frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}.$$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{f_{uu} \cdot f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}.$$

$$H = \frac{lG - 2mF + nE}{2(EG-F^2)} = \frac{(1+f_v^2) f_{uu} - 2 f_u f_v f_{uv} + (1+f_u^2) f_{vv}}{2(1+f_u^2+f_v^2)^{\frac{3}{2}}}.$$

$H = 0$ iff $(1+f_v^2) f_{uu} - 2 f_{uv} f_{uv} + (1+f_u^2) f_{vv} = 0.$

In our case, $f_v = h'$, $f_{uu} = g''$, $f_u = g'$, $f_{uv} = 0$, $f_{vv} = h''$.

$$\Rightarrow \quad (1+h'^2)g'' + (1+g'^2)h'' = 0.$$

$$\Rightarrow \quad \frac{-g''(u)}{1+g'(u)^2} = \frac{h''(v)}{1+h'(v)^2} = a$$

$$-\int \frac{dg'}{1+g'^2} = au + c$$

$$\arctan(g') = -au + c \qquad \text{Case (a) } a \neq 0.$$

For simplicity, $c=0$, $au \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$g' = \tan(-au)$$

$$g = \frac{1}{a}\ln(\cos au) + c$$

Similarly, $h = -\frac{1}{a}\ln(\cos av) + \tilde{c}$

$$f(u,v) = g(u) + h(v) = \frac{1}{a}\ln\left(\frac{\cos au}{\cos av}\right) + c + \tilde{c}.$$

Case (b) $a=0$. Then $g''(u) = h''(v) = 0$

$\Rightarrow$ g, h are linear functions.

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② Geodesics

Let $X(u^1, u^2)$ be a coordinate patch of M, $\alpha(t)$ be any reg curve on M (may not be parametrized by arc-length). Then $\alpha(t)$ may be represented by $X(u^1(t), u^2(t))$.

$$\alpha' = X_i \frac{du^i}{dt}, \quad \alpha'' = X_i \left(\frac{d^2u^i}{dt^2}\right) + X_{ij} \frac{du^i}{dt}\frac{du^j}{dt}$$

Recall $X_{ij} = \Gamma^k_{ij} X_k + h_{ij} U$. Then

$$\alpha'' = \frac{d^2u^k}{dt^2} X_k + \Gamma^k_{ij} \frac{du^i}{dt}\frac{du^j}{dt} X_k + h_{ij}\frac{du^i}{dt}\frac{du^j}{dt} U$$

$$= \left(\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right) X_k + h_{ij}\frac{du^i}{dt}\frac{du^j}{dt} U$$

$$\triangleq \underbrace{}_{(\alpha'')_{tan}} + \underbrace{}_{(\alpha'')_{nor}}.$$

We say a regular curve $\alpha(t)$ in M is a geodesic of M if $(\alpha'')_{tan} = 0$.

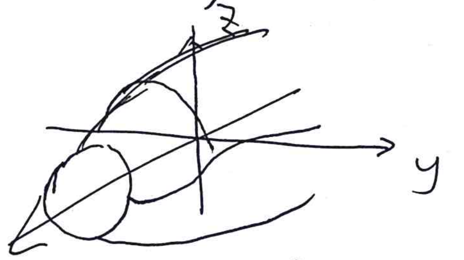
A obvious property for geodesic : $|\alpha'| = $ constant.

$\left(\frac{d}{dt}\langle \alpha', \alpha' \rangle = 2\langle \alpha'', \alpha' \rangle = 2\langle (\alpha'')_{tan}, \alpha' \rangle = 0. \right)$

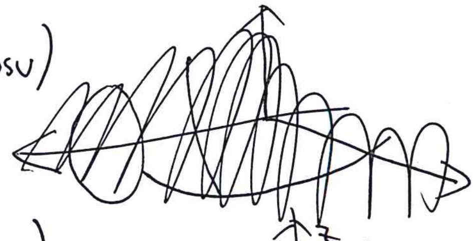
Let's ~~consider~~ study geodesics in some surfaces.

(a) surface of revolution (about X-axis)

$$X(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

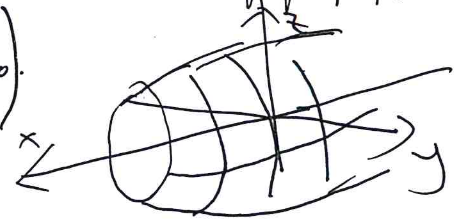


$$X_u = (g',\ h'\cos v,\ h'\sin v),\quad X_v = (0,\ -h\sin v,\ h\cos v)$$



$$X_{uu} = (g'',\ h''\cos v,\ h''\sin v)$$

$\alpha(u) = X(u, v_0)$ = meridian (for some fixed v_0).



$$(\alpha'')_{tan} = \langle X_u, X_{uu}\rangle X_u + \langle X_v, X_{uu}\rangle X_v$$

$$= (g''\cdot g' + h''\cdot h')\,X_u$$

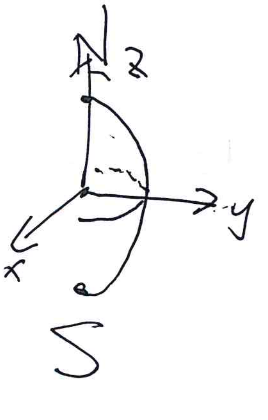
$$= \frac{1}{2}(g'^2 + h'^2)'\,X_u$$

$\alpha(u)$ is a geodesic iff $(g'^2+h'^2)' = 0$ iff $\alpha(u)$ has constant speed

(a1) geodesics in S^2_R = sphere with radius R.

$$X(u,v) = (R\cos u\cos v,\ R\sin u\cos v,\ R\sin v)$$

(note: z-axis = axis of revolution
a meridian $X(u_0,v) = (R\cos u_0\cos v,\ \sin u_0 R\cos v,\ R\sin v)$ = longitude)



$g(v) = R\sin v$, $h(v) = R\cos v$

$g'^2 + h'^2 = R^2$ = constant $\Rightarrow$ $X(u_0, v)$ is a geodesic $\forall\ u_0$ fixed.

Symmetry $\Rightarrow$ great circles are geodesics.

$\Downarrow$

$S^2_R \cap$ plane passing thro. the centre of S^2_R.

Conversely, if $\alpha(s)$ is a geodesic parametrized by arc-length,

then
$$\alpha'' = \cancel{(\alpha'')_{tan}}^{\,0} + \langle \alpha'', U \rangle U$$

$$U = \frac{\alpha(t)}{R} \quad \Rightarrow \quad \alpha'' = \langle \alpha'', \alpha \rangle \frac{\alpha}{R^2}$$

$$(U \times T)' = \left(\frac{\alpha \times \alpha'}{R} \right)'$$
$$= \frac{\alpha \times \alpha''}{R}$$
$$= \frac{\alpha \times \langle \alpha'', \alpha \rangle \frac{\alpha}{R^2}}{R}$$
$$= 0.$$

$\Rightarrow$ $U \times T \triangleq \widehat{N}$ is a constant vector.

$$\langle \widehat{N}, \alpha \rangle = \langle U \times T, \alpha \rangle = \left\langle \frac{\alpha \times \alpha'}{R}, \alpha \right\rangle = 0$$

$\Rightarrow$ α belongs to the plane with normal $\widehat{N}$

Since $\langle \underline{U}, \hat{N} \rangle = 0$, $\underline{U}$ is also in that plane

i.e the plane contains the centre of S^2_R

$\Rightarrow \alpha$ belongs to a great circle.

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By our def. of geodesic, given a coordinate patch $X(u_1, u_2)$.

α = geodesic iff $\frac{d^2u^k}{dt^2} + \Gamma^k_{ij} \frac{du^i}{dt}\frac{du^j}{dt} = 0 \quad \forall k = 1,2$

This is a non-linear 2^{nd} order ODE. By ODE's theory, given initial data $(u^1(0), u^2(0))$, $\left(\frac{d}{dt}u^1\Big|_{t=0}, \frac{d}{dt}u^2\Big|_{t=0}\right)$, we have a unique short time ~~existence~~ solution. We call the above equations geodesic equation.

(b) Geodesics in cylinder.

$X(\theta, z) = (\cos\theta, \sin\theta, z) \quad \theta \in [0, 2\pi), z \in \mathbb{R}$

One may compute its 1^{st} F.F. and Christoffel symbol at any pt

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^k_{ij} = 0 \quad \forall i,j,k = 1,2.$$

Then the geodesic equation is

$$\begin{cases} \dfrac{d^2\theta}{dt^2} = 0 \\ \dfrac{d^2 z}{dt^2} = 0 \end{cases}$$

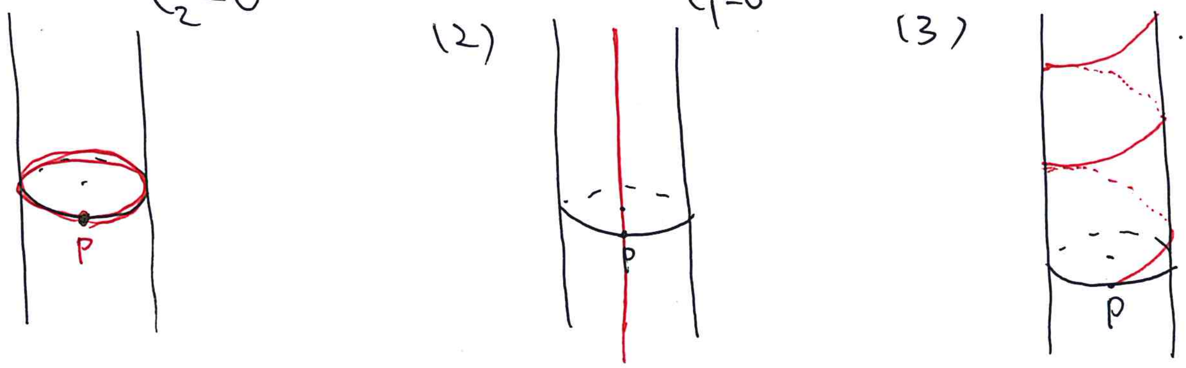
Initial data: $p = (1,0,0)$, $V_p = c_1 X_\theta\big|_p + c_2 X_z\big|_p$

where c_1, c_2 are any fix real number.

$$\Rightarrow \begin{cases} \theta(t) = c_1 t + 2k\pi \\ z(t) = c_2 t \end{cases} \quad \text{for some } k \in \mathbb{Z}.$$

Actually, there are 3 kinds of geodesics.

(1) $c_2 = 0$ (2) $c_1 = 0$ (3) $c_1 \neq 0$ and $c_2 \neq 0$.



Ex: Does there exist a surface $X(u,v)$ with

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix} \text{ and } (h_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix} ?$$

Ex 2: Show that if M is a cpt, connected, oriente constant mean curvature surface with positive Gauss curvature, then M is a sphere.