Suggested Sol. of Assignment 6

(1) Gauss curvature $K = \dfrac{-4}{(1+u^2+v^2)^4}$.

$$H = \frac{1}{2}\,\frac{LG - 2mF + nE}{EG - F^2}, \qquad \mathrm{II} \triangleq \begin{pmatrix} L & m \\ m & n \end{pmatrix}$$

$$= \frac{1}{2}\,\frac{-2\cdot(r^2+1)^2 + 2\cdot(r^2+1)^2}{(1+r^2)^4}, \qquad \begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$= 0.$$

$$k_1 = H + \sqrt{H^2 - K} = \frac{2}{(1+u^2+v^2)^2}$$

$$k_2 = H - \sqrt{H^2 - K} = \frac{-2}{(1+u^2+v^2)^2}.$$

(2) The surface is represented by

$$X(t,\theta) = \left(\sin t\cos\theta,\ \sin t\sin\theta,\ \cos t + \log\tan\frac{t}{2}\right)$$

for $t \in (0, \frac{\pi}{2})$, $\theta \in [0, 2\pi)$.

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}}$$

~~$\det \mathrm{I} = \sin^4 t + \sin^2 t + 1 - 2\sin^2 t(\cos^2\theta + \sin^2\theta)$~~

~~$\det \mathrm{I}$~~ denote $a = \sin t$, $b = \cos t$, $c = \sin\theta$, $d = \cos\theta$.

$$X_t = (bd, bc, -a + \tfrac{1}{a}), \quad X_\theta = (-ac, ad, 0)$$

$$X_t \times X_\theta = (-b^2 d, -b^2 c, ab) \qquad \det \mathrm{I} = b^2$$

$$h_{tt} = -\frac{b}{a}, \quad h_{t\theta} = 0, \quad h_{\theta\theta} = ab$$

$$\Rightarrow K = -1.$$

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(3) (i) Let $\alpha(t) = (x(t), y(t), z(t))$, $n(t) = (n_1(t), n_2(t), n_3(t))$ (along α

$$d(fn)(V_1) \overset{\Delta}{=} \frac{d}{dt}\Big|_{t=0} (fn_1, fn_2, fn_3)$$

$$= \langle \nabla f, V_1 \rangle n - f S(V_1)$$

$$d(fn)(V_2) = \langle \nabla f, V_2 \rangle n - f S(V_2)$$

$$\langle d(fn)(V_1) \times d(fn)(V_2), n \rangle = f^2 \langle S(V_1) \times S(V_2), n \rangle$$

To show $K = \dfrac{\langle d(fn)(V_1) \times d(fn)(V_2), n \rangle}{f^2}$, it

suffices to show $K_p = \langle S_p(V_1) \times S_p(V_2), n \rangle$. $(*)$

Note that $n = V_1 \times V_2$ and $\det \mathrm{I} = 1$ (at p since V_1, V_2 are o.

$$\det \mathrm{II} = \langle S_p(V_1), V_1 \rangle \langle S_p(V_2), V_2 \rangle - \langle S_p(V_1), V_2 \rangle^2$$

So we get $(*)$ by Lagrange's identity and symmetry of S_p

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(ii) $n = \dfrac{\nabla h}{|\nabla h|}$, note that $|\nabla h| = 2f$

So $d(fn)(v) = \left(\dfrac{v_1}{a^2}, \dfrac{v_2}{b^2}, \dfrac{v_3}{c^2}\right)$ if $v = (v_1, v_2, v_3)$,

~~Then~~ and $n = \dfrac{1}{f}\left(\dfrac{x}{a^2}, \dfrac{y}{b^2}, \dfrac{z}{c^2}\right)$.

$$K = \frac{\left\langle \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2}\right) \times \left(\frac{w_1}{a^2}, \frac{w_2}{b^2}, \frac{w_3}{c^2}\right), \frac{1}{f}\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) \right\rangle}{f^2} \qquad (*)$$

where $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$ with $v \times w = n$.

$$(*) = \frac{\left\langle \left(a^2(v_2 w_3 - v_3 w_2), b^2(v_3 w_1 - v_1 w_3), c^2(v_1 w_2 - w_1 v_2)\right), \left(\frac{x}{a^2}, \frac{y}{b^2}, \right.\right.}{f^3 a^2 b^2 c^2}$$

Note that $\sim = \frac{1}{f}(x, y, z)$ Since $v \times w = f\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$.

And note that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

We have $(*) = \frac{1}{f^4 a^2 b^2 c^2}$. #