Suggested Solution of Midterm of Differential Geometry

(1)(a)

Proof. We denote the unit tangent, principal normal and binormal of α by T, N, B respectively. Then we have

$$\begin{aligned}\alpha' &= |\alpha'|T \\ \alpha'' &= |\alpha'|'T + |\alpha'|T' \\ &= |\alpha'|'T + |\alpha'|^2 kN\end{aligned}$$

by Frenet Formula for non-unit speed curve. Then

$$\alpha' \times \alpha'' = k|\alpha'|^3 T \times N = k|\alpha'|^3 B.$$

Thus, we have

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

□

(b)

$$\begin{aligned}\alpha(t) &= (t, \frac{1+t}{t}, \frac{1-t^2}{t}) \\ \alpha' &= (1, \frac{-1}{t^2}, -\frac{1}{t^2} - 1), |\alpha'| = \sqrt{2}\sqrt{1 + \frac{1}{t^2} + \frac{1}{t^4}} \\ \alpha'' &= 2(0, \frac{1}{t^3}, \frac{1}{t^3}) \\ \alpha' \times \alpha'' &= \frac{1}{t^3}(1, -1, 1), |\alpha' \times \alpha''| = 2\sqrt{3}\frac{1}{|t|^3} \\ k &= \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{\sqrt{6}}{2}\frac{1}{(t^2 + 1 + \frac{1}{t^2})^{\frac{3}{2}}}.\end{aligned}$$

(2)

Proof. Let C be the circle such that AB is a chord of it and AB divide C into α and β where α has length l (we also denote the length of β by l'). Such a circle exists because $l > AB$. If r is the radius and θ is the angle subtended by AB, then $r\theta = l$ and $r\sin(\theta/2) = \frac{1}{2}AB$. The area of the circle is πr^2. Note that $l + l' = 2\pi r$.

Consider any curve γ with length l joining A, B such that $\{\gamma, AB\}$ forms a Jordan curve. Then we consider the closed curve $\{\gamma, \beta\}$ with fixed length $l+l'$, and let γ vary, β be fixed. By a general isoperimetric inequality, let $4\pi^2 r^2 = (l + l')^2 \geq 4\pi\mathcal{A}$ where $\mathcal{A}$ is the area bounded by the curve $\{\gamma, \beta\}$. Therefore the largest possible area is πr^2. Hecne if $\{\gamma, \beta\}$ bounds the largest area, we must have $(l + l')^2 \geq 4\pi\mathcal{A}$ and $\{\gamma, \beta\}$ must be a circle.

□

(3)(a)

Proof. Fix s_0, and let A be a constant orthogonal matrix and $\mathbf{v}_0$ be a constant vector such that:

$$\begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} = A \begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix}$$

and $\alpha(s_0) = A\beta(s_0) + \mathbf{v}_0$. Let $\gamma(s) = A\beta(s) + \mathbf{v}_0$. Then one can check that the curvature and torsion of γ are still k and τ respectively. By the Frenet formula, the $T_\gamma, N_\gamma, B_\gamma$ and T_α, N_α, B_a satisfy the same ODE and they are equal at $s = s_0$. Hence they are equal for all s. In particular, $\alpha'(s) = \gamma'(s)$. Since $\alpha(s_0) = \gamma(s_0)$, we have $\alpha(s) = A\beta(s) + \mathbf{v}_0$.

□

(3)(b)

Proof. Let $R = aT + bN + cB$, then by the assumption

$$\begin{cases} T' = R \times T \\ N' = R \times N \\ B' = R \times B \end{cases}$$

we have

$$\begin{cases} bN \times T + cB \times T = T' = kN \\ aT \times N + cB \times N = N' = -kT + \tau B \\ aT \times B + bN \times B = B' = -\tau N \end{cases}$$

One may apply inner product with N, T to the above equations, then we have $c = k, b = 0, a = \tau$. Therefore, $R = \tau T + kB$. □

(4)(a) In the coordinate parametrization

$$X(u, v) = (\frac{4u}{A}, \frac{4v}{A}, \frac{2(u^2 + v^2)}{A}),$$

we have

$$X_u = (\frac{4(v^2 - u^2 + 4)}{A^2}, \frac{-8uv}{A^2}, \frac{16u}{A^2})$$
$$X_v = (\frac{-8uv}{A^2}, \frac{4(u^2 - v^2 + 4)}{A^2}, \frac{16v}{A^2})$$
$$E = \frac{16}{(u^2 + v^2 + 4)^2}, F = 0, G = \frac{16}{(u^2 + v^2 + 4)^2}.$$

And

$$\begin{aligned} Area(\mathbb{S}^2) &= \int_{\mathbb{R}^2} \sqrt{EG - F^2} dudv \\ &= \int_{\mathbb{R}^2} \frac{16}{(u^2 + v^2 + 4)^2} dudv \\ &= \int_0^\infty \int_0^{2\pi} \frac{16}{(r^2 + 4)^2} r dr d\theta \\ &= 2\pi \int_0^\infty \frac{16}{(r^2 + 4)^2} r dr \\ &= 4\pi. \end{aligned}$$

(5)(a)

Proof. Let $X : U \subset \mathbb{R}^2 \to M$ be a parametrization of p in M, and write $X(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in U$. Since $X_u \times X_v \neq 0$, we have one of the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(u, v)}, \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}$$

is not zero at $q \doteq X^{-1}(p)$.

Suppose first that $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$, and consider the map $\pi \circ X : U \to \mathbb{R}^2$ where π is the projection $\pi(x, y, z) = (x, y)$. Then $\pi \circ X(u, v) = (x(u, v), y(u, v))$. Then by inverse function theorem, there exist an open neighborhood of q, V_1, in U and an open neighborhood of $\pi \circ X(q)$, V_2, such that $F = \pi \circ X$ maps V_1 diffeomorphically onto V_2. Now let $V = X(V_1)$, then V is an open neighborhood of p in M. Let $Y = X \circ F^{-1}$. Then Y is a coordinate parametrization near p. Now

$$Y = (x(u(x, y), v(x, y)), y(u(x, y), v(x, y)), z(u(x, y), v(x, y)) = (x, y, f(x, y)).$$

by the definition of F^{-1}. □

(5)(b)

Proof. Now we have

$$\begin{aligned} X(u,v) =&(x(u,v), y(u,v), f(x(u,v), y(u,v))) \\ X_u =&(x_u, y_u, f_x x_u + f_y y_u) \\ X_v =&(x_v, y_v, f_x x_v + f_y y_v). \end{aligned}$$

Then (x_u, y_u) and (x_v, y_v) are linearly independent otherwise X_u and X_v are linearly dependent. By inverse function theorem, there are an open neighborhood O of (u_0, v_0) and an open neighborhood V of (x_0, y_0) such that $\pi \circ X : O \to V$ is a diffeomorphism.

□

(5)(c)

Proof. Let $p \in M$, $X : U \to M,(u,v) \mapsto (x,y,z)$ and $Y : V \to M,(\xi,\eta) \mapsto (x,y,z)$ be 2 coordinate parametrizations such that $p \in X(U) \cap Y(V) \doteq S$. Denote $U_1 = X^{-1}(S)$ and $V_1 = Y^{-1}(S)$, we want to show $Y^{-1} \circ X : U_1 \to V_1$ is a diffeomorphism. Now let $p' \in S$, by part (a) and (b)(we may assume the graph we obtain in part (a) is over xy-plane), we have $(u,v) \mapsto (x,y)$ is diffeomorphic near $X^{-1}(p')$. Similarly, $(\xi,\eta) \mapsto (x,y)$ is diffeomorphic near $Y^{-1}(p')$. So $(u,v) \mapsto (\xi,\eta)$ is diffeomorphic near $X^{-1}(p')$. We have shown that $Y^{-1} \circ X : U_1 \to V_1$ is a local diffeomorphism. Together with the fact $Y^{-1} \circ X$ is bijective, we actually see $Y^{-1} \circ X$ is a diffeomorphism.

□