1. Geodesic

Recall: Let α be regular curve on an orientable regular surface M with unit normal vector field $\mathbf{n}$. Suppose α is parametrized by arc length, then the geodesic curvature k_g is given by $k_g = \langle \alpha'', \mathbf{n} \times \alpha' \rangle$. Note that if we change the orientation by using $-\mathbf{n}$ instead, then the geodesic will be changed to $-k_g$.

Definition 1. A regular curve on a regular surface M is called a *geodesic* if it is parametrized proportional to arc length and has zero geodesic curvature.

Being geodesic (i.e. $k_g = 0$, with $|\alpha'|$=constant) does not depend on orientation.

Proposition 1. *Let α be a regular curve on a regular surface M parametrized proportional to arc length. α is a geodesic if and only if $(\alpha'')^T = 0$, where $(\alpha'')^T$ is the projection onto the tangent plane of M.*

2. Geodesic curvature in local coordinates

Let M be a regular surface and $\mathbf{X}(u_1, u_2)$ be a coordinate parametrization. Let $\mathbf{n} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$.

Lemma 1. *Let $\alpha(t)$ be a regular curve on M such that $\alpha(t) = \mathbf{X}(u_1(t), u_2(t))$. Then*

$$\alpha'' = \sum_{k=1}^{2} \mathbf{X}_k \left(u_k'' + \sum_{i,j=1}^{2} \Gamma_{ij}^k u_i' u_j' \right) + c\mathbf{n}$$

for some smooth function c.

Question: What is c?

Lemma 2. *Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}^3$, then*

$$\langle \mathbf{u}_1 \times \mathbf{u}_2, \mathbf{v}_1 \times \mathbf{v}_2 \rangle = \langle \mathbf{u}_1, \mathbf{v}_1 \rangle \langle \mathbf{u}_2, \mathbf{v}_2 \rangle - \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \langle \mathbf{u}_2, \mathbf{v}_2 \rangle.$$

Proposition 2. *Geodesic curvature is intrinsic. In fact, if α is parametrized by arc length, then*

$$k_g = \sqrt{\det(g_{ij})} \Big[u_1' u_2'' - u_2' u_1'' + \Gamma_{11}^2 (u_1')^3 - \Gamma_{22}^1 (u_2')^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1\right) (u_1')^2 u_2' - \left(2\Gamma_{12}^1 - \Gamma_{22}^2\right) (u_2')^2 u_1' \Big]$$

Sketch of proof:

$$
\begin{aligned}
k_g =& \frac{\langle \alpha' \times \alpha'', \mathbf{X}_1 \times \mathbf{X}_2 \rangle}{\sqrt{\det(g_{ij})}} \\
=& \frac{1}{\sqrt{\det(g_{ij})}} \left(\langle \alpha', \mathbf{X}_1 \rangle \langle \alpha'', \mathbf{X}_2 \rangle - \langle \alpha', \mathbf{X}_2 \rangle \langle \alpha'', \mathbf{X}_1 \rangle \right) \\
=& \sqrt{\det(g_{ij})} \left[u_1' \left(u_2'' + \sum_{i,j=1}^{2} \Gamma_{ij}^2 u_i' u_j' \right) - u_2' \left(u_1'' + \sum_{i,j=1}^{2} \Gamma_{ij}^1 u_i' u_j' \right) \right] \\
=& \sqrt{\det(g_{ij})} \Big[u_1' u_2'' - u_2' u_1'' + \Gamma_{11}^2 (u_1')^3 - \Gamma_{22}^1 (u_2')^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1 \right) (u_1')^2 u_2' \\
& - \left(2\Gamma_{12}^1 - \Gamma_{22}^2 \right) (u_2')^2 u_1' \Big]
\end{aligned}
$$

□

Corollary 1. *Isometry will carry geodesics to geodesics.*

Proposition 3. *Suppose t is arc length (or proportional to arc length), then the geodesic curvature is zero if and only if*

$$
u_k'' + \sum_{i,j=1}^{2} \Gamma_{ij}^k u_i' u_j' = 0 \tag{1}
$$

for $k = 1, 2$.

Lemma 3. *Suppose $\alpha(t)$ is a regular curve on M which satisfies (1) in any coordinate chart. Then $|\alpha'|$ is constant.*

Proposition 4. *A smooth curve on M is a geodesic if and only if it satisfies (1) in any coordinate chart.*

Proposition 5. *At any point $p \in M$, and any vector $\mathbf{v} \in T_p(M)$, there is a geodesic $\alpha(t)$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.*

Appendix

Theorem 1. *Let U be an open set in $\mathbb{R}^n$ and let $I_a = (-a, a) \subset \mathbb{R}$, with $a > 0$. Suppose $\mathbf{F} : U \times I_a \to \mathbb{R}^n$ is a smooth map. Then for any $\mathbf{x}_0 \in U$, there is $0 < \delta < a$, such that the following IVP has a solution:*

$$
\begin{cases} \mathbf{x}'(t) = & \mathbf{F}(x(t), t), \ -\delta < t < \delta; \\ \mathbf{x}(0) = & \mathbf{x}_0. \end{cases}
$$

Moreover, the solutions of the IVP is unique. Namely, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the above IVP on $(-b, b)$ for some $0 < b < a$, then $\mathbf{x}_1 = \mathbf{x}_2$.

Assignment 8, Due Friday Nov 14, 2014

(1) (a) Find the absolute value of the curvature of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points $(a, 0)$ and $(0, b)$. Assuming $a, b > 0$.

(b) Intersect the cylinder $C = \{(x, y, z)|x^2 + y^2 = 1\}$ with a plane passing through the x-axis and making an angle θ with the xy-plane. Show that the curve α is an ellipse. Also find the absolute value of the geodesic curvature of α at the points where α meets their axes (i.e. major and minor axes of the ellipse).

(2) Let $\alpha(\tau)$ be a regular curve on a regular surface M, where τ may not be proportional to arc length. Let $\alpha' = \frac{\partial \alpha}{\partial \tau}$, etc. Prove that α is a geodesic after reparametrization if and only if $(\alpha'')^T = \lambda(\tau)\alpha'$ for some smooth function τ on α.

(*Hint:* If $(\alpha'')^T = \lambda(\tau)\alpha'$, then reparametrize the curve by t, with $t = \int_a^\tau \exp(\int_a^\rho \lambda(s)ds)d\rho$ so that $\frac{d^2t}{d\tau^2} = \lambda \frac{dt}{d\tau}$.)

(3) Suppose M is a connected orientable regular surface such that the principal curvature is a constant. That is: there is a constant λ such that the principal curvatures at *every point* are equal to λ. Prove that M is contained in a plane or in a sphere.

(*Note*: This result together with a result in previous exercise implies that if all points on M are umbilical, then M is contained in a plane or a sphere. This face is used to prove that a compact surface in $\mathbb{R}^3$ with constant Gaussian curvature must be a sphere.)