1. Gauss map and Gaussian curvature

Let M be an orientable regular surface and let $\mathbf{n}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{n}$. That is $\mathbf{n}: M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$.

Proposition 1. *Let $p \in M$. Suppose $K(p) \neq 0$. Let B_n be a sequence of open sets with $B_n \to p$ in the sense that $\sup_{q\in B_n} |p-q| \to 0$ as $n \to \infty$. Let A_n be the area of B_n and $\widetilde{A}_n$ be the area of the Gauss image $\mathbf{n}(B_n)$ of B_n. Then*

$$\lim_{n\to\infty} \frac{\widetilde{A}_n}{A_n} = |K(p)|.$$

Proof. Let $\mathbf{X}(u,v)$ be a coordinate paramatrization near p such that $\mathbf{X}(0,0) = p$. Let $S_p(\mathbf{X}_u) = a_{11}\mathbf{X}_u + a_{21}\mathbf{X}_v$ and $S_p(\mathbf{X}_u) = a_{12}\mathbf{X}_u + a_{22}\mathbf{X}_v$, then $\det(a_{ij}) = K$. Suppose U_n in the (u,v) plane such that $\mathbf{X}(U_n) = B_n$. Then

$$A_n = \iint_{U_n} |\mathbf{X}_u \times \mathbf{X}_v| dudv,$$

and

$$\widetilde{A}_n = \iint_{U_n} |\mathbf{n}_u \times \mathbf{n}_v| dudv.$$

where $\mathbf{n} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ is the unit normal on M. Now

$$\mathbf{n}_u = d\mathbf{n}(\mathbf{X}_u) = -S_p(\mathbf{X}_u) = -(a_{12}\mathbf{X}_u + a_{22}\mathbf{X}_v), \mathbf{n}_v = -(a_{12}\mathbf{X}_u + a_{22}\mathbf{X}_v).$$

Hence

$$\mathbf{n}_u \times \mathbf{n}_v = \det(a_{ij})\mathbf{X}_u \times \mathbf{X}_v = K\mathbf{X}_u \times \mathbf{X}_v.$$

So

$$\begin{aligned} |\mathbf{n}_u \times \mathbf{n}_v|(u,v) =& |K(u,v)||\mathbf{X}_u \times \mathbf{X}_v|(u,v) \\ =& |K(0,0)||\mathbf{X}_u \times \mathbf{X}_v|(u,v) + (|K(u,v) - |K(0,0)|)\, |\mathbf{X}_u \times \mathbf{X}_v|(u,v). \end{aligned}$$

Since $B_n \to p$, we have $U_n \to (0,0)$. Hence for any $\epsilon > 0$ there is $N > 0$ such that if $n \geq N$, then $||K(u,v) - |K(0,0)|| \leq \epsilon$.

$$\widetilde{A}_n = |K(0,0)|A_n + R_n$$

where $|R_n| \leq \epsilon A_n$, if $n \geq N$. From this it is easy to see the proposition follows. □

2. Theorema Egregium of Gauss

Definition 1. Let $F : M_1 \to M_2$ be a diffeomorphism. F is said to be an *isometry* if for any $p \in M_1$ and $q = F(p)$, the linear map $dF : M_1 \to M_2$ is an isometry as inner product spaces. If there is an isometry from M_1 onto M_2, then M_1 is said to be isometric to M_2.

Theorem 1. *(Theorema Egregium of Gauss) The Guassian curvature K is invariant under isometries. That is to say, the Gaussian curvature depends only on the first fundamental form.*

1st Proof: sketch. Let $\mathbf{X}(u,v)$ be a local parametrization of a regular surface, and let $E, F,\ G$ be the coefficients of the first fundamental form and $e,\ f,\ g$ be the second fundamental form. In the following, if $a,\ b,\ c$ are three vectors, (a,b,c) is the ordered triple product of the three vectors. Now

$$e = \langle \mathbf{n}, \mathbf{X}_{uu} \rangle = \frac{(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v)}{\sqrt{EG - F^2}},$$

etc.

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{[(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v)(\mathbf{X}_{vv}, \mathbf{X}_u, \mathbf{X}_v) - (\mathbf{X}_{uv}, \mathbf{X}_u, \mathbf{X}_v)^2]}{(EG - F^2)^2}. \end{aligned}$$

Hence

$$\begin{aligned}
&(EG - F^2)^2 K \\
&= \det(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v) \det(\mathbf{X}_{vv}, \mathbf{X}_u, \mathbf{X}_v) - (\det(\mathbf{X}_{uv}, \mathbf{X}_u, \mathbf{X}_v))^2 \\
&= \begin{vmatrix} \langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix} - \begin{vmatrix} \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix} \\
&= \begin{vmatrix} \langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix} \\
&\quad - \begin{vmatrix} 0 & \langle \mathbf{X}_{uv}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix}
\end{aligned}$$

Now

$$\begin{aligned}
\langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle &= \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle_v - \langle \mathbf{X}_{uuv}, \mathbf{X}_v \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle_u + \langle \mathbf{X}_{uvu}, \mathbf{X}_v \rangle \\
&= \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle_v - \frac{1}{2} G_{uu} \\
&= \langle \mathbf{X}_u, \mathbf{X}_v \rangle_{uv} - \langle \mathbf{X}_u, \mathbf{X}_{vu} \rangle_v - \frac{1}{2} G_{uu} \\
&= F_{uv} - \frac{1}{2} E_{vv} - \frac{1}{2} G_{uu}.
\end{aligned}$$

Hence we have

$$K = \frac{A - B}{(EG - F^2)^2}$$

where

$$A=\begin{vmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix},$$

and

$$B=\begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}.$$

□

2nd Proof: sketch. Let us use the following notations: $\mathbf{X}(u_1,u_2)$ is a coordinate parametrization. Let $\mathbf{X}_i=\mathbf{X}_{u_i}$, $g_{ij}=\langle\mathbf{X}_i,\mathbf{X}_j\rangle$, $(g^{ij})=(g_{ij})^{-1}$, $\mathbf{n}=\mathbf{X}_1\times\mathbf{X}_2/|\mathbf{X}_1\times\mathbf{X}_2|$. Let (a_{ij}) be the matrix of $-d\mathbf{n}$ w.r.t. the basis $\{\mathbf{X}_1,\mathbf{X}_2\}$ and $b_{ij}=\mathbb{II}(\mathbf{X}_i,\mathbf{X}_j)=\langle\mathbf{X}_{ij},\mathbf{n}\rangle$.

Then

$$\mathbf{X}_{ij}=b_{ij}\mathbf{n}+\Gamma_{ij}^k\mathbf{X}_k$$

Note $\Gamma_{ij}^k=\Gamma_{ji}^k$. *Einstein summation convention: repeated indices mean summation.*

Hence

$$\begin{aligned}\mathbf{X}_{ijm}=&b_{ij,m}\mathbf{n}+b_{ij}\mathbf{n}_m+\Gamma_{ij,m}^k\mathbf{X}_k+\Gamma_{ij}^k\mathbf{X}_{km}\\ =&b_{ij,m}\mathbf{n}+b_{ij}\left(-a_{mk}\mathbf{X}_k\right)+\Gamma_{ij,m}^k\mathbf{X}_k+\Gamma_{ij}^k\left(b_{km}\mathbf{n}+\Gamma_{km}^l\mathbf{X}_l\right)\\ =&\left(b_{ij,m}+\Gamma_{ij}^kb_{km}\right)\mathbf{n}+\left(-b_{ij}a_{km}+\Gamma_{ij,m}^k+\Gamma_{ij}^s\Gamma_{sm}^k\right)\mathbf{X}_k\end{aligned}$$

Since $\mathbf{X}_{112}=\mathbf{X}_{121}$, we have

$$\left(-b_{11}a_{k2}+\Gamma_{11,2}^k+\Gamma_{11}^s\Gamma_{s2}^k\right)\mathbf{X}_k=\left(-b_{12}a_{k1}+\Gamma_{12,1}^k+\Gamma_{12}^s\Gamma_{s1}^k\right)\mathbf{X}_k.$$

So

$$-b_{11}a_{22}+\Gamma_{11,2}^2+\Gamma_{11}^s\Gamma_{s2}^2=-b_{12}a_{21}+\Gamma_{12,1}^2+\Gamma_{12}^s\Gamma_{s1}^2. \tag{1}$$

Now

$$(a_{ij})=(g^{ij})(b_{ij}).$$

Hence

$$a_{22}=g^{2i}b_{i2}=\frac{1}{\det(g_{ij})}\left(-g_{12}b_{12}+g_{11}b_{22}\right),$$

$$a_{21}=g^{2i}b_{i1}=\frac{1}{\det(g_{ij})}\left(-g_{12}b_{11}+g_{11}b_{21}\right).$$

So (1) becomes:

$$-b_{11}\frac{1}{\det(g_{ij})}\left(-g_{12}b_{12}+g_{22}b_{22}\right)+\Gamma^2_{11,2}+\Gamma^s_{11}\Gamma^2_{s2}$$
$$=-b_{12}\frac{1}{\det(g_{ij})}\left(-g_{12}b_{11}+g_{22}b_{21}\right)+\Gamma^2_{12,1}+\Gamma^s_{12}\Gamma^2_{s1}.$$

Hence

$$g_{11}K=g_{11}\frac{\det(b_{ij})}{\det(g_{ij})}=\Gamma^2_{11,2}-\Gamma^2_{12,1}+\Gamma^s_{11}\Gamma^2_{s2}-\Gamma^s_{12}\Gamma^2_{s1}$$

and

$$K=\frac{1}{g_{11}}\left(\Gamma^2_{11,2}-\Gamma^2_{12,1}+\Gamma^s_{11}\Gamma^2_{s2}-\Gamma^s_{12}\Gamma^2_{s1}\right).$$

The theorem follows from the following lemma. □

Lemma 1.

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{l=1}^{2}g^{kl}\left(g_{il,j}+g_{jl,i}-g_{ij,l}\right).$$

where $g_{ij,l}=\frac{\partial}{\partial u_l}g_{ij}$ *etc.*

Sketch of proof.

$$\begin{aligned}g_{km}\Gamma^k_{ij}=&\langle\Gamma^k_{ij}\mathbf{X}_k,\mathbf{X}_m\rangle\\=&\langle\mathbf{X}_{ij},\mathbf{X}_m\rangle\\=&g_{im,j}-\langle\mathbf{X}_i,\mathbf{X}_{mj}\rangle\\=&g_{im,j}-g_{ik}\Gamma^k_{mj}.\end{aligned}$$

Hence

(2) $$g_{im,j}=g_{km}\Gamma^k_{ij}+g_{ik}\Gamma^k_{mj}$$

Permuting i,m,j, we also have

(3) $$g_{ij,m}=g_{kj}\Gamma^k_{im}+g_{ik}\Gamma^k_{jm}$$

(4) $$g_{jm,i}=g_{km}\Gamma^k_{ji}+g_{jk}\Gamma^k_{mi}.$$

(1) + (3) − (2) gives:

$$2g_{km}\Gamma^k_{ij}=g_{im,j}+g_{jm,i}-g_{ij,m}.$$

Multiple by g^{lm} and sum on m, we get the result. □

Assignment 7, Due Friday Nov 7, 2014

(1) Prove that if $\mathbf{X}$ is an orthogonal parametrization, i.e. $F = 0$, then the Gaussian curvature is given by:

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Suppose in addition $E = G$ everywhere, then

$$K = -e^{-2f}\Delta f$$

where f is such that $E = e^{2f}$ (i.e. $f = \frac{1}{2}\log E$), and Δ is the Laplacian operator:

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}.$$

(2) (i) Let M be a regular orientable surface, such that M is tangent to a sphere $\mathbb{S}^2(r)$ of radius r at p. Assume that p is at the origin and $T_p(M)$ is the xy-plane. Suppose also that M lies inside $\mathbb{S}^2(r)$ which lies in the upper half space $\{z \geq 0\}$. Prove that the Gaussian curvature of M at p is at least $1/r^2$.

(ii) Prove that a compact regular orientable surface in $\mathbb{R}^3$ contains a point with positive Gaussian curvature.

(3) Let M be a regular orientable surface. Suppose every point of M is umbilical, i.e., there is a smooth function λ on M such that for any $p \in M$, $S_p(\mathbf{v}) = \lambda(p)\mathbf{v}$ for all $\mathbf{v} \in T_p(M)$. Suppose M is connected (i.e. any two points on M can be joined by a continuous curve on M), then λ is constant on M.