Gaussian curvature and mean curvature

Let M be a regular surface which is orientable. Let $\mathbf{n}$ be a smooth unit normal vector field on M.

Definition 1. Let $p \in M$ and let S_p be the shape operator.

(i) The determinant of S_p is called the *Gaussian curvature* of M at p and is denoted by $K(p)$.
(ii) $\frac{1}{2}\mathrm{tr}(S_p)$ is called the *mean curvature* of M at p and is denoted by $H(p)$.

Proposition 1. *Let M be a regular surface which is orientable. Let $\mathbf{n}$ be a smooth unit normal vector field on M. Let $p \in M$ and let k_1 and k_2 be the principal curvatures of M at p (i.e. eigenvalues of S_p). Then $K(p) = k_1 k_2$ and $H(p) = \frac{1}{2}(k_1 + k_2)$.*

Gaussian curvature and mean curvature in local coordinates

Let $\mathbf{X}(u, v)$ be a coordinate parametrization of M. Let $\mathbf{n} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$.

Definition 2. The coefficients of the second fundamental form e, f, g at p are defined as:

$$\begin{aligned} e =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u) = \langle S_p(\mathbf{X}_u), \mathbf{X}_u \rangle = -\langle d\mathbf{n}(\mathbf{X}_u), \mathbf{X}_u \rangle = \langle \mathbf{n}, \mathbf{X}_{uu} \rangle; \\ f =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_v) = \langle S_p(\mathbf{X}_u), \mathbf{X}_v \rangle = -\langle d\mathbf{n}(\mathbf{X}_u), \mathbf{X}_v \rangle = \langle \mathbf{n}, \mathbf{X}_{uv} \rangle; \\ g =& \mathbb{II}_p(\mathbf{X}_v, \mathbf{X}_v) = \langle S_p(\mathbf{X}_v), \mathbf{X}_v \rangle = -\langle d\mathbf{n}(\mathbf{X}_v), \mathbf{X}_v \rangle = \langle \mathbf{n}, \mathbf{X}_{vv} \rangle. \end{aligned}$$

Question: Why is it true that $\langle S_p(\mathbf{X}_u), \mathbf{X}_u \rangle = \langle \mathbf{n}, \mathbf{X}_{uu} \rangle$ etc?
Fact: Note that

$$e = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})}{\sqrt{EG - F^2}}, f = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})}{\sqrt{EG - F^2}}, g = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv})}{\sqrt{EG - F^2}}.$$

Proposition 2. (1) *Let*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be the matrix of S_p with respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v\}$. Then

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

(2) *The Gaussian curvature $K(p)$ and the mean curvature $H(p)$ are given by*

$$K(p) = \frac{eg - f^2}{EG - F^2},$$

and

$$H(p) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Facts:

(i) K and H are smooth functions on M.

(ii) $k_1 = H + \sqrt{H^2 - K}$, $k_2 = H - \sqrt{H^2 - K}$. (Note $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$). Hence if $k_1 > k_2$, then k_1 and k_2 are also smooth.

Definition 3. Let M be a regular orientable surface. Let $p \in M$. p is said to be: (i) elliptic point if $K(p) > 0$; (ii) hyperbolic point if $K(p) < 0$; (iii) parabolic point if $K(p) = 0$ but $S_p \neq 0$; (iv) planar point if $S_p = 0$.

Proposition 3. *Let M be a regular orientable surface. Let $p \in M$. If $K(p) > 0$ then near p, points on M will be on one side of $T_p(M)$. If $K(p) < 0$, then in each neighborhood of p there exist points on both sides of $T_p(M)$.*

Assignment 6, Due Friday Oct 31, 2014

(1) Find the Gaussian curvature of the Enneper's surface:

$$\mathbf{X}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, v^2 - u^2\right).$$

Show that the mean curvature of the Enneper's surface is 0 everywhere. What are the principal curvatures?

(2) Consider the tractrix Let $\alpha : (0, \frac{\pi}{2}) \to xz$-plane given by

$$\alpha(t) = \left(\sin t, 0, \cos t + \log \tan \frac{t}{2}\right).$$

Show that the Gaussian curvature of the surface of revolution obtained by rotating α about the z-axis is -1. The surface is called the pseudosphere.

(3) Let M be a regular orientable surface with unit normal vector field $\mathbf{n}$. Let f be a smooth function on M which is nowhere zero. Let $p \in M$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $T_p(M)$.

(i) Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{n})(\mathbf{v}_1) \times d(f\mathbf{n})(\mathbf{v}_2), \mathbf{n}\rangle}{f^2}.$$

Note $d(f\mathbf{n})(\mathbf{v})$ is defined as follow: let α be the curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{n})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{n}(\alpha(t)))\big|_{t=0}.$$

(ii) Let M be the ellipsoid

$$h(x, y, z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let f be the restriction of the function

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}.$$

Apply (i) to show that the Gaussian curvature is given by

$$K = \frac{1}{f^4 a^2 b^2 c^2}.$$

(Hint: We may take $\mathbf{n} = \frac{\nabla h}{|\nabla h|}$. Note that $|\nabla h| = 2f$ and so $d(f\mathbf{n})(\mathbf{v}) = \left(\frac{v_1}{a^2}, \frac{v_2}{v^2}, \frac{v_3}{c^2}\right)$ if $\mathbf{v} = (v_1, v_2, v_3)$.)