## The second fundamental form and curvatures of curves

Let $M$ be a regular surface which is orientable. Let $\mathbf{n}$ be a smooth unit normal vector field on $M$. (Note: There are two choices of unit normal vector fields on an orientable surface). Let $\alpha(s)$ be a smooth curve on $M$. Let $T = \alpha'$ and let $\tilde{N}(s)$ be the unit vector at $\alpha(s)$ such that $\tilde{N}(s) \in T_{\alpha(s)}(M)$ and such that $\{T, \tilde{N}, \mathbf{n}\}$ is positively oriented, i.e. $\tilde{N} = \mathbf{n} \times T$.

**Lemma 1.** *$T'$ is a linear combination of $\tilde{N}$ and $\mathbf{n}$. Hence $T' = k_g\tilde{N} + k_n\mathbf{n}$ for some smooth functions $k_n$ and $k_g$ on $\alpha(s)$.*

**Definition 1.** As in the lemma, $k_n(s)$ is called the *normal curvature* of $\alpha$ at $\alpha(s)$ and $k_g(s)$ is called the *geodesic curvature* of $\alpha$ at $\alpha(s)$.

**Facts:**

(i) $k_n$ and $k_g$ depend on the choice of $\mathbf{n}$.

(ii) We will see later that $k_g$ is intrinsic: it depends only on the first fundamental form *and* the orientation of the surface.

(iii) Let $k$ be the curvature of $\alpha'$. Suppose $k$ is not zero. Let $N$ be the normal of $\alpha$. Then $k_n = k\langle N, \mathbf{n}\rangle = k\cos\theta$ where $\theta$ is the angle between $N$ and $\mathbf{n}$. If $k = 0$, then $T' = 0$ and $k_n = k_g = 0$.

For the time being we only discuss normal curvature. The geometric meaning of the second fundamental form is the following:

**Proposition 1.** *Let $M$ be an orientable regular surface and $\mathbf{n}$ be a smooth unit normal vector field on $M$. Let $\mathrm{I\!I}$ be the second fundamental form of $M$ (w.r.t. $\mathbf{n}$) and let $p \in M$. Suppose $\mathbf{v} \in T_p(M)$ with unit length and suppose $\alpha(s)$ is a smooth curve of $M$ parametrized by arclength with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. Then*

$$k_n(0) = \mathrm{I\!I}_p(\mathbf{v}, \mathbf{v})$$

*where $k_n$ is the normal curvature of $\alpha$ at $\alpha(0) = p$.*

**Corollary 1.** *With the same notation as in the proposition, we have the following:*

(i) *Let $\alpha$ and $\beta$ be two regular curves parametrized by arc length passing through $p$. Suppose $\alpha$ and $\beta$ are tangent at $p$. Then the normal curvatures of $\alpha$ and $\beta$ at $p$ are equal.*

(ii) *Let $k_1$ and $k_2$ be the eigenvalues of $\mathrm{I\!I}_p$ with $k_1 \le k_2$. Then all normal curvatures are between $k_1$ and $k_2$.*

**Definition 2.** $k_1$ and $k_2$ in the corollary are called *principal curvatures* of $M$ at $p$. Let $e_1$ and $e_2$ be eigenvectors of $k_1$ and $k_2$. The directions determined by $e_1$ and $e_2$ are called *principal directions.*

More definitions:

**Definition 3. 1.** *Line of curvature*: A regular curve $\alpha$ on $S$ such that $\alpha'$ is an eigenvector of the 2nd fundamental form.

**2.** *Asymptotic direction and asymptotic curve*: The direction where the normal curvature is zero is called an asymptotic direction. A regular curve $\alpha$ on $S$ such that $\alpha'$ is in the asymptotic direction is called an asymptotic curve.

**3.** *Conjugate directions*: Two $\mathbf{v}, \mathbf{w} \in T_p(S)$ are conjugate if they are not zero vectors and $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = 0$. The directions parallel to $\mathbf{v}$ and $\mathbf{w}$ are said to be conjugate directions.

### More on bilinear forms

Let $V$ be a two dimensional inner product space with inner product $\langle \cdot \rangle$. Let $B$ be a symmetric bilinear form on $V$ and let $A$ be the corresponding self adjoint linear operator on $V$. Let $k_1$ and $k_2$ be the eigenvalues of $A$.

**Proposition 2.** (i) $\det A = k_1 k_2$ *and* $\mathrm{tr}(A) = k_1 + k_2$.

(ii) *Let* $\mathbf{v}_1$, $\mathbf{v}_2$ *form a basis for* $V$ *(not necessarily orthonormal). Let* $b_{ij} = B(\mathbf{v}_i, \mathbf{v}_j)$ *and let* $g_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. *Then* $\det(A) = \frac{\det(b_{ij})}{\det(g_{ij})}$ *and* $\mathrm{tr}(A) = \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{\det(g_{ij})}$.

### Assignment 5, Due Friday Oct 24, 2014

(1) Let $M = \{(x, y, z) | \ z = x^2 + ky^2\}$, with $k > 0$. Show that $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ form a basis of $T_p(M)$ where $p = (0, 0, 0)$. Let $\mathbf{n}$ be the unit normal of $M$ pointing upward, i.e. $\langle \mathbf{n}, e_3 \rangle > 0$ where $e_3 = (0, 0, 1)$. Find the matrix of $d\mathbf{n}_p : T_p(M) \to T_p(M)$ with respect to the basis $e_1$, $e_2$. Find the principal curvatures of $M$ at $p$.

(2) (Euler formula) Let $M$ be an orientable regular surface with a unit normal vector field $\mathbf{n}$. Let $p \in M$ and let $k_1$, $k_2$ be the principal curvatures with eigenvectors $e_1$ and $e_2$ which are orthonormal. Let $\mathbf{v} \in T_p(M)$ such that $\mathbf{v} = \cos\theta e_1 + \sin\theta e_2$. Prove that the normal curvature of the curve $\alpha$ passing through $p$ with tangent vector $v$ is given by

$$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

(3) The total torsion of a regular curve $\alpha : [0, l] \to \mathbb{R}^3$, which has nonzero curvature and is parametrized by arc length, is given by $\int_0^l \tau ds$, where $\tau$ is the torsion of $\alpha$. Show that a smooth closed curved with nowhere zero curvature on a sphere of radius $R$ has zero total torsion.