The second fundamental form

Let M be a regular surface, which is **orientable**. Let $\mathbf{n}$ be a unit normal vector field on M. (Note that $\mathbf{n}$ must be smooth). Let $p \in M$, and $\mathbf{v} \in T_p(M)$ and let $\alpha(t)$ be a smooth curve on M with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.

Lemma 1. *With the above notation,* $\frac{d}{dt}(\mathbf{n}(\alpha(t))|_{t=0}$ *depends only on* $\mathbf{v}$ *and does not depend on* α*.*

$\frac{d}{dt}(\mathbf{n}(\alpha(t))|_{t=0}$ in the lemma will be denoted by $d\mathbf{n}(\mathbf{v})$.

Lemma 2. *With notation as in the previous lemma,* $d\mathbf{n}(\mathbf{v})$ *is tangent to* M*. Moreover, if* $\mathbf{v}, \mathbf{w} \in T_p(M)$*, then* $\langle d\mathbf{n}(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{v}, d\mathbf{n}(\mathbf{w})\rangle$*. Hence the map* $S_p(\mathbf{v}) = -d\mathbf{n}(\mathbf{v})$ *is a self-adjoint linear map on* $T_p(M)$*, and* $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = \langle S_p(\mathbf{v}), \mathbf{w})\rangle$ *is a symmetric bilinear form on* $T_p(M)$*.*

Definition 1. $S_p : T_p(M) \to T_p(M)$ is called the *shape operator or Weingarten map* of M at p.

Definition 2. $\mathbb{II}_p$ is called the *second fundamental form* of M at p.

Definition 3.
(i) Let M be regular surface and let $f : M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X} : U \to M$.
(ii) M_1, M_2 be regular surfaces and let $F : M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ F \circ \mathbf{X}$ is smooth whenever it is defined.

Let $F : M_1 \to M_2$ be a smooth map. Let $p \in M$ and $q = F(p)$. For any $\mathbf{v} \in T_p(M_1)$, let $\alpha(t)$ be a smooth curve on M_1 with $\alpha(0) = p$ and $\alpha'(o) = \mathbf{v}$. Define $dF_p(\mathbf{v}) = \frac{d}{dt}F(\alpha(t))|_{t=0}$.

Lemma 3. dF_p *is well-defined and is a linear map from* $T_p(M_1) \to T_q(M_2)$*.* dF_p *is called the differential of* F *at* p*.*

Definition 4. Let M be an orientable regular surface and let $\mathbf{n}$ be a smooth unit normal vector field on M. Then the map:

$$\mathbf{n} : M \to \mathbb{S}^2 \subset \mathbb{R}^3$$

which assigns each point $p \in M$ to the unit normal $\mathbf{n}(p)$ at p is called the Gauss map of M.

Question: Is there only one Gauss map for M?

Facts: (i) $\mathbf{n}$ is a smooth map from M to $\mathbb{S}^2$; (ii) if $p \in M$ and $q = \mathbf{n}(p)$, then $T_p(M)$ and $T_q(\mathbb{S}^2)$ are the same vector subspaces of $\mathbb{R}^3$.

(iii) The differential of $\mathbf{n}$ is the negative of the shape operator if we identify $T_p(M)$ with $T_q(\mathbb{S}^2)$, where $q = \mathbf{n}(p)$.

A fact on symmetric bilinear form

Theorem 1. *Let* $(V, \langle\,\rangle)$ *be a finite dimensional inner product space of dimension* n *and let* B *be a symmetric bilinear form. Then there is an orthonormal basis* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *such that* B *is diagonalized. Namely,* $B(\mathbf{v}_i, \mathbf{v}_j) = \lambda_i \delta_{ij}$.

Proof. We just prove the case that $n = 2$. Let S be the set in V with $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. Then $B(\mathbf{v}, \mathbf{v})$ attains maximum on S at some $\mathbf{v}$. Let $\mathbf{v}_1 \in S$ be such that

$$B(\mathbf{v}_1, \mathbf{v}_1) = \max_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v}).$$

Let $\mathbf{v}_2 \in S$ such that $\mathbf{v}_1 \perp \mathbf{v}_2$. It is sufficient to prove that $B(\mathbf{v}_1, \mathbf{v}_2) = 0$. Let $t \in \mathbb{R}$ and let

$$f(t) = \frac{B(\mathbf{v}_1 + t\mathbf{v}_2, \mathbf{v}_1 + t\mathbf{v}_2)}{||\mathbf{v}_1 + t\mathbf{v}_2||^2}.$$

Then $f'(0) = 0$. Hence

$$\begin{aligned} 0 =& 2B(\mathbf{v}_1, \mathbf{v}_2) - 2B(\mathbf{v}_1, \mathbf{v}_1)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \\ =& 2B(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

□

Note that if A is the self-adjoint linear map such that $B(\mathbf{v}, \mathbf{w}) = \langle A\mathbf{v}, \mathbf{w} \rangle$. Then the $\mathbf{v}_i$ in the theorem are eigenvectors of A with eigenvalues λ_i.