****Reminder: Midterm exam will be on Oct 8, Wednesday 10:30am–12:15pm****

A proof on change of coordinates

Proposition 1. *(Change of coordinates) Let M be a regular surface and let $\mathbf{X} : U \to M$, $\mathbf{Y} : V \to M$ be two coordinate parametrizations. Let $S = \mathbf{X}(U) \cap \mathbf{Y}(V) \subset M$ and let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$. Then $\mathbf{Y}^{-1} \circ \mathbf{X} : U_1 \to V_1$ is a diffeomorphism.*

Proof. (Sketch) Let $p \in S$. Then there is an open set $S_1 \subset S$ such that S_1 is given by the graph $\{(x, y, z) | (x, y) \in \mathcal{O}, z = f(x, y)\}$. Now if $(u, v) \in U_1$ with $\mathbf{X}(u, v) \in S_1$, then

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v))$$

because $z = f(x, y)$. Then

$$\mathbf{X}_u = (x_u, y_u, f_x x_u + f_y y_u), \mathbf{X}_v = (x_v, y_v, f_x x_v + f_y y_v).$$

Since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we have $(x_u, y_u), (x_v, y_v)$ are linearly independent (why?). This implies $(u, v) \to (x, y)$ is diffeomorphic near $\mathbf{X}^{-1}(p)$. Similarly, if $(\xi, \eta) \in V_1$, then $(\xi, \eta) \to (x, y)$ is diffeomorphic near $\mathbf{Y}^{-1}(p)$. Hence $(\xi, \eta) \to (u, v)$ is diffeomorphic. $\square$

Regular parametrized surface

Definition 1. Let U be an open set of $\mathbb{R}^2 = \{(u, v)\}$ and let $\mathbf{X} : U \to \mathbb{R}^3$ be a smooth map. Suppose $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent at every point in U. Then $\mathbf{X}(U)$ is called a *regular parametrized surface.*

A regular parametrized surface locally is a regular surface.

Surfaces of revolution

Let $\alpha(v) = (f(v), 0, g(v))$ be a regular curve of the xz-plane. The surface by rotating α around the z-axis is given by

$$\mathbf{X}(u, v) = (f(v) \cos u, f(v) \sin u, g(v)).$$

Then $\mathbf{X}_u = (-f(v) \sin u, f(v) \cos u, 0)$, $\mathbf{X}_v = (f'(v) \cos u, f'(v) \sin u, g'(v))$. Note that $\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0$. They are linearly independent if $f(v) \neq 0$.

Ruled surfaces

Proposition 2. *Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular parametrized surface. For any $p = (u_0, v_0) \in U$, there is a neighborhood V of p such that $\mathbf{X}(V)$ is a regular surface.*

Ruled surfaces are in general regular parametrized surfaces. They are defined as follows.

Let $\alpha(t)$ and $\mathbf{w}(t)$, $t \in I$ be two curves. Then

$$\mathbf{X}(t,v) = \alpha(t) + v\mathbf{w}(t) \tag{1}$$

$t \in I$, $v \in \mathbb{R}$ is called a *ruled surface*. In this case, we say that the ruled surface is generated by $\{\alpha, \mathbf{w}\}$. $\mathbf{X}_t = \alpha' + v\mathbf{w}', \mathbf{X}_v = \mathbf{w}$.

Here are some terms:

Directrix: $\alpha(t)$.

Rulings: $L_t = \{t = \text{constant}\}$, is the straight line $v \to \alpha(t) + v\mathbf{w}(\mathbf{t})$, $v \in \mathbb{R}$.

The Möbius strip

$$\mathbf{X}(\theta, v) = (\cos\theta, \sin\theta, 0) + v(\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta) = \alpha(\theta) + v\mathbf{w}(\theta)$$

$-\pi < \theta < \pi$, $-\frac{1}{2} < v < \frac{1}{2}$. When $\theta = \pi$, $\mathbf{w}(\theta) = (-1, 0, 0)$. When $\theta = -\pi$, then $\mathbf{w}(\theta) = (1, 0, 0)$. Now

$$\mathbf{X}_v = (\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)$$

$$\mathbf{X}_\theta = (-\sin\theta, \cos\theta, 0) + v\mathbf{w}'(\theta)$$

Tangent space of a regular surface

Let M be a regular surface and let $p \in M$. Let $\mathbf{X} : U \to M$ be a coordinate chart containing p with $\mathbf{X}(u_0, v_0) = p$. Then the *tangent space* $T_p(M)$ *of* M *at* p is the vector space spanned by $\mathbf{X}_u(u_0, v_0), \mathbf{X}_v(u_0, v_0)$. Note that $\dim(T_p(M)) = 2$.

Proposition 3. (i) *$T_p(M)$ is well defined. Namely, the definition of $T_p(M)$ does not depend on the coordinate chart containing p.*

(ii) *$\vec{v} \in T_p(M)$ if any only if there is a smooth curve α on M with $\alpha(0) = p$ and $\alpha'(0) = \vec{v}$.*

Lemma 1. *Let $\mathbf{X} : U \to V \subset M$ be a coordinate parametrization of a regular surface M. Let $\beta(t) = (u(t), v(t), t \in I$ be a smooth curve in U, then $\alpha(t) = \mathbf{X}(u(t), v(t))$ is a smooth curve in M. Conversely, suppose $\alpha(t)$ is smooth curve in M with $\alpha \subset V$, then there is a unique smooth curve $\beta(t)$ in U such that $\alpha(t) = \mathbf{X}(\beta(t))$.*

First fundamental form

Let M be a regular surface. The *first fundamental form* g of M is an inner product at each $T_p(M)$ given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$.

Let $\mathbf{X} : U \to V \subset M$ be a coordinate parametrization. The coefficients of the first fundamental form g are defined as:

$$E = g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_u, \mathbf{X}_u \rangle, F = g(\mathbf{X}_u, \mathbf{X}_v) = \langle \mathbf{X}_u, \mathbf{X}_v \rangle, G = g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_v, \mathbf{X}_v \rangle.$$

If we use (u^1, u^2) instead of (u, v) and let $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, then we also denote coefficients of the first fundamental form g as

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M, $a \le t \le b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t))$ in local coordinates. Then the length of α is given by

$$\begin{aligned} l &= \int_a^b |\alpha'|(t) dt \\ &= \int_a^b \sqrt{E(\alpha(t))(\frac{du}{dt})^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))(\frac{dv}{dt})^2} dt \\ &= \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} dt. \end{aligned}$$

So sometimes, the first fundamental form is written symbolically as $Edu^2 + 2Fdudv + Gdv^2$.

Bilinear form and quadratic form

Let V be a vector space over $\mathbb{R}$. A map $B : V \times V \to \mathbb{R}$ is said to be a bilinear form if

$$B(a_1\mathbf{u}_1 + a_2\mathbf{u}_2, \mathbf{v}) = a_1 B(\mathbf{u}_1, \mathbf{v}) + a_2 B(\mathbf{u}_2, \mathbf{v});\ B(\mathbf{u}, b_1\mathbf{v}_1 + b_2\mathbf{v}_2) = b_1 B(\mathbf{u}, \mathbf{v}_1) + b_2 B(\mathbf{u}, \mathbf{v}_2).$$

B is said to be symmetric if $B(\mathbf{u}, \mathbf{v}) = B(\mathbf{v}, \mathbf{u})$. If B is a symmetric bilinear form, then $Q(\mathbf{u}) = B(\mathbf{u}, \mathbf{u})$ is called the quadratic form associated with B.

Proposition 4. (i) *Let B be a symmetric bilinear form on V and Q is the associated quadratic form. Then*

$$B(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left[Q(\mathbf{u} + \mathbf{v}) - Q(\mathbf{u}) - Q(\mathbf{v})\right].$$

(ii) *Let B be a bilinear symmetric form on a* **finite dimensional inner product space** *V with dimension n and Q is the associated quadratic form. Then there is an orthonormal basis $\{e_1, \ldots, e_n\}$ such that $B(e_i, e_j) = \lambda_i \delta_{ij}$ and $Q(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n \lambda_i a_i^2$. λ_i are called the eigenvalues of B (or Q).*

(iii) *Let B, V and Q as in (ii) and let $\langle ; \rangle$ denote the inner product on V. The the mapping $A : V \to V$ with $A\mathbf{u}$ defined by*

$$\langle A\mathbf{u}, \mathbf{v} \rangle = B(\mathbf{u}, \mathbf{v})$$

for all $\mathbf{v}$ is a self-adjoint linear transformation. Conversely, every self-adjoint linear transformation A induces a symmetric bilinear form B by the above relation. Moreover, the λ_i in (ii) are the eigenvalues of A.

Assignment 4, Due Friday Oct 3, 2014

(1) Find the equation of the tangent space:

(i) at a point (a, b, c) on the regular surface $M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = 0\}$, assuming 0 is a regular value of f;

(ii) at the point $(a, b, 0)$ on the surface $x^2 + y^2 - z^2 = 1$.

Find a unit normal in (i) and (ii).

(2) Find the coefficients E, F, G of first fundamental forms of the following surfaces:

(i) $\mathbf{X}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$, $a, b, c > 0$; (ellipsoid)

(ii) the unit sphere under the stereographic projection:

$$x(u, v) = \frac{4u}{A}, y(u, v) = \frac{4v}{A}, z(u, v) = \frac{2(u^2 + v^2)}{A},$$

where $A = u^2 + v^2 + 4$.

(3) Let $\mathbf{X}(u, v)$ be a coordinate parametrization of a regular surface. Prove that $|\mathbf{X}_u \times \mathbf{X}_v| = \sqrt{EG - F^2}$.

(4) Let C be a simple (i.e. no self intersection) regular curve in xz-plane with length l and is given by $\alpha(s) = (x(s), 0, z(s))$ with $0 \le s \le l$ is the arc length. Assume $x(s) > 0$ for all s. Show that the area of the surface M of revolution by rotating C about the z axis is given by

$$Area(M) = 2\pi \int_0^l x(s) ds.$$