Regular surfaces

Definition 1. A subset $M \subset \mathbb{R}^3$ is said to be a *regular surface* if for any $p \in M$, there is an open neighborhood U of p in M, an open set D in $\mathbb{R}^2$ and a map $\mathbf{X} : D \to M \cap U$ such that the following are true:

(rs1) $\mathbf{X}$ is smooth.

(rs2) $\mathbf{X}$ is full rank: $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{X}_v = \frac{\partial \mathbf{x}}{\partial v}$ are linearly independent, for any $(u, v) \in D$.

(rs3) $\mathbf{X}$ is a homeomorphism from D onto $M \cap U$. (That is: $\mathbf{X}$ is bijective, $\mathbf{X}$ and $\mathbf{X}^{-1}$ are continuous).

Let M be a regular surface, a map $\mathbf{x} : U \to V$ where V is an open set of M, satisfying the above conditions. $\mathbf{X}$ is called a *parametrization (a system of local coordinates)*, and V is called a *coordinate chart (patch)*. If $\mathbf{X}(u, v) = p$, then (u, v) are called local coordinates of p.

Basic properties on coordinate chart

We need the inverse function theorem. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open set U to $\mathbb{R}^n$, $F(\mathbf{x}) = \mathbf{y}(\mathbf{x}) =$ where $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{y} = (y^1, \ldots, y^m)$. Let $\mathbf{x_0} = (x_0^1, \ldots, x_0^n) \in U$. The Jacobian matrix of F at $\mathbf{x}_0$ is the $m \times n$ matrix

$$dF_{\mathbf{x}_0} = \left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0) \right).$$

Theorem 1. (Inverse Function Theorem) *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose $F(\mathbf{x}_0) = \mathbf{y}_0$ and $dF_{\mathbf{x}_0}$ is nonsingular. Then there exist open sets $U \supset V \ni \mathbf{x}_0$ and $W \ni \mathbf{y}_0$, such that F is a diffeomorphism from V to W. That is to say, $F : V \to W$ is bijective and F^{-1} is also smooth on W.*

Proposition 1. *Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(x) = a$, then $\nabla f(x) \neq \mathbf{0}$.) Then*

$$M = \{x \in U | \ f(x) = a\}$$

is a regular surface.

Proposition 2. *Let M be regular surface and let $\mathbf{X} : U \to M$ be a coordinate parametrization. Then for any $p = (u_0, v_0) \in U$ there is a open set $V \subset U$ with $p \in V$ such that $\mathbf{X}(V)$ is a graph over an open set in one of the coordinate plane.*

Proposition 3. *(Change of coordinates) Let M be a regular surface and let $\mathbf{X} : U \to M$, $\mathbf{Y} : V \to M$ be two coordinate parametrizations. Let $S = \mathbf{X}(U) \cap \mathbf{Y}(V) \subset M$ and let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$. Then $\mathbf{Y}^{-1} \circ \mathbf{X} : U_1 \to V_1$ is a diffeomorphism.*

Proposition 4. *Suppose M is a regular surface and suppose $\mathbf{X} : U \to M$ with U being an open set in $\mathbb{R}^2$ such that $\mathbf{X}$ satisfies (rs1) and (rs2). Suppose $\mathbf{X}$ is one to one, then $\mathbf{X}$ also satisfies (rs3). Hence $\mathbf{X} : U \to M$ is a coordinate parametrization.*

Definition 2. (i) Let M be regular surface and let $f : M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X} : U \to M$.

(ii) M_1, M_2 be regular surfaces and let $F : M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ \mathbf{X}$ is smooth whenever it is defined.

Note that a parametrized curve on M is defined as a curve $\alpha : I \to \mathbb{R}^3$ such that $\alpha(t) \in M$ for all t.

Assignment 3, Due Friday, 26/9/2014

(1) Show that the hyperboloid of one sheet:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
is a regular surface and the following is a parametrization:
$$\mathbf{X}(u, v) = (a \cosh u \cos v, b \cosh u \sin v, c \sinh u).$$
Find the domain on (u, v) plane such that $\mathbf{X}$ is one to one.

(2) Let $\mathbf{S}^1$ be the unit circle $x^2 + y^2 = 1$. Let $\alpha(s), 0 \le s \le 2\pi$, be a parametrization of $\mathbf{S}^1$ by arc length. Let $\mathbf{w}(s) = \alpha'(s) + e_3$ where $e_3 = (0, 0, 1)$. Show the ruled surface
$$\mathbf{X}(s, v) = \alpha(s) + v\mathbf{w}(s)$$
with $-\infty < v < \infty$, is the hyperboloid $x^2 + y^2 - z^2 = 1$. Is $\mathbf{X}$ a surjective map to the hyperboloid? Is $\mathbf{X}$ injective? Does $\mathbf{X}$ has rank 2 for $0 < s < 2\pi$, $v \in \mathbb{R}$?

(3) Find coordinate charts which cover the catenoid obtained by revolving the catenary $y = \cosh x$ about the x-axis.

(4) The Enneper's surface is defined by
$$\mathbf{X}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$
Show that this a regular surface for $u^2 + v^2 < 3$. Also find two points on the circle $u^2 + v^2 = 3$ such that they have the same image under $\mathbf{X}$.