****Midterm exam will be on Oct 8, Wednesday 10:30am–12:15pm****

Some properties on curves

Theorem 1. *Let α be a regular curves in $\mathbb{R}^3$ parametrized by arc length.*

(i) *Suppose the curvature $k \equiv 0$ if and only if α is a straight line.*
(ii) *Suppose the curvature $k > 0$ and the torsion $\tau \equiv 0$ if and only if α is a plane curve.*
(iii) *Suppose the curvature $k = k_0 > 0$ is a constant and $\tau \equiv 0$, then α is a circular arc with radius $1/k_0$.*
(iv) *Suppose the curvature $k > 0$ and the torsion $\tau \neq 0$ everywhere. α lies on s sphere if and only if $\rho^2 + (\rho')^2\lambda^2 =$constant, where $\rho = 1/k$ and $\lambda = 1/\tau$.*
(v) *Suppose the curvature $k = k_0 > 0$ is a constant and $\tau = \tau_0$ is a constant. Then α is a circular helix.*
(vi) *Suppose α is defined on $[a, b]$. Let $\mathbf{p} = \alpha(a)$ and $\mathbf{q} = \alpha(b)$. Then the length l of α satisfies $l \geq |\mathbf{p} - \mathbf{q}|$. Moreover, equality holds if and only if α is the straight line from $\mathbf{p}$ to $\mathbf{q}$.*

Plane curves

Definition 1. Let $\alpha : [a, b] \to \mathbb{R}^2$ be a regular plane curve. Let T be the unit tangent of α. Define the unit normal of α to be the unit vector N such that T, N is positively oriented. Define the curvature k to be the number such that $T' = kN$.

Isoperimetric inequality

Definition 2. Let $\alpha : [a, b] \to \mathbb{R}^3$ be a regular smooth or piecewise smooth curved.

(i) α is said to be **closed** if α is smooth at a, b and $\alpha(a) = \alpha(b)$, $\alpha'(a) = \alpha'(b), \ldots$.
(ii) α is said to be **simple** if $\alpha(t_1) \neq \alpha(t_2)$ for $a \leq t_1 < t_2 < b$.

Theorem 2. (Jordan curve theorem) *Every simple closed curve on $\mathbb{R}^2$ will divide $\mathbb{R}^2$ into two arcwise connected components, one of them is bounded (the interior of the curve) and another one is unbounded (the exterior of the curve).*

Definition 3. Let $\alpha : [a, b] \to \mathbb{R}^2$ be a closed curve which bounds a domain D. α is said to be *positively oriented* if the unit normal N is pointing into the interior of α.

Theorem 3. (Isoperimetric inequality) *Let α be a simple closed curve on $\mathbb{R}^2$. Let A be the area of the interior of α and l be the length of α. Then $l^2 \geq 4\pi A$, with equality holds if and only if α is a circle.*

We need the *Divergence Theorem, or Gauss-Green's Theorem or Ostrogradsky's Theorem*:

Theorem 4. *Let* $\mathbf{X}$ *be a smooth vector field defined and in the interior of a positively oriented closed curve* α *which bounds a domain* D *in* $\mathbb{R}^2$*.*

$$\int_D \operatorname{div}\mathbf{X}\, dxdy = \int_\alpha \langle \mathbf{X}, \nu\rangle ds$$

where ν *is the unit outward normal of* α*.*

We also need the following fact from Fourier series.

Lemma 1. *Let* f *be a smooth periodic function on* $\mathbb{R}$ *with period* 2π *such that*

$$\int_0^{2\pi} f(t)dt = 0.$$

Then

$$\int_0^{2\pi} (f')^2(t)dt \ge \int_0^{2\pi} f^2(t)dt$$

with equality holds if and only if $f(t) = a\cos t + b\sin t$ *for some constants* a, b*.*

Assignment 2: Due Friday, 19/9/2014

(1) Suppose α is regular curve with curvature $k(s) = k_0 > 0$ is constant and torsion $\tau(s) = \tau_0 \neq 0$ is constant. Show that α is part of a circular helix and determine the circular helix in terms of k_0 and τ_0.
(2) Let α be a regular curve parametrized by arc length with curvature $k > 0$. Let T be the unit tangent vector of α. Prove that τ/k is constant if and only if there is a constant unit vector $\vec{u}$ such that $\langle T, \vec{u}\rangle$ is constant on α. (Such a curve is called a generalized helix).
(3) Let AB be line segment in $\mathbb{R}^2$. Let $l >$ length of AB be fixed. Show that the curve α joining A and B, with length l, which together with AB forms a Jordan curve, bounds the largest possible area is an arc of a circle passing through A and B.
(4) Let α be a smooth plane curve in $\mathbb{R}^2$ which is contained in a disk of radius $r > 0$. Prove that there is a point $p \in \alpha$ such that the curvature $k(p)$ of α at p satisfies $|k(p)| \ge 1/r$.