A proof of the lemma in previous note

Lemma 1. *Let $a_1(t), a_2(t)$ be smooth functions on $(T_1, T_2) \subset \mathbb{R}$ such that $a_1^2 + a_2^2 = 1$. For any $t_0 \in (T_1, T_2)$ and θ_0 such that $a_1(t_0) = \cos\theta_0$, $a_2(t_0) = \sin\theta_0$, there exists unique a smooth function $\theta(t)$ with $\theta(t_0) = \theta_0$ such that $a_1(t) = \cos\theta(t)$ and $a_2(t) = \sin\theta(t)$.*

Proof. (Sketch) Suppose θ satisfies the condition. Then $a_1' = -\theta'\sin\theta$, $a_2' = \theta'\cos\theta$. Hence $\theta' = a_1a_2' - a_2a_1'$. From this we have uniqueness. To prove existnce, fix $t_0 \in (T_1, T_2)$ and let θ_0 be such that $\cos\theta_0 = a_1(0)$, $\sin\theta_0 = a_2(0)$. Let

$$\theta(t) = \theta_0 + \int_{t_0}^{t} (a_2'a_1 - a_1'a_2)d\tau.$$

Let $f = (a_1 - b_1)^2 + (a_2 - b_2)^2$, where $b_1 = \cos\theta, b_2 = \sin\theta$. Then $f = 2 - 2a_1b_1 - 2a_2b_2$. Then

$$\begin{aligned}
-\frac{1}{2}f' =& a_1'b_1 + a_1b_1' + a_2'b_2 + a_2b_2' \\
=& a_1'b_1 - \theta'a_1b_2 + a_2'b_2 + \theta'a_2b_1 \\
=& (a_2'a_1 - a_1'a_2)(-a_1b_2 + a_2b_1) + a_1'b_1 + a_2'b_2 \\
=& -a_1^2a_2'b_2 + a_2a_2'a_1b_1 + a_1a_1'a_2b_2 - a_2^2a_1'b_1 + a_1'b_1 + a_2'b_2 \\
=& -a_1^2a_2'b_2 - a_1a_1'a_1b_1 - a_2a_2'a_2b_2 - a_2^2a_1'b_1 + a_1'b_1 + a_2'b_2 \\
=& 0
\end{aligned}$$

because $a_1^2 + a_2^2 = 1$ and $a_1a_1' + a_2a_2' = 0$. □

Minimal surfaces

Definition 1. A regular surface M is said to be *minimal* if the mean curvature of M is identically zero.

Definition 2. Let $\mathbf{X}(u, v)$ be a local parametrization of a regular surface. $\mathbf{X}$ is said to be *isothermal* if $|\mathbf{X}_u| = |\mathbf{X}_v|$, and $\langle \mathbf{X}_u, \mathbf{X}_v\rangle = 0$.

To check whether a surface is minimal, the following fact is useful.

Proposition 1. *Let $\mathbf{X}(u, v)$ be an isothermal coordinate parametrization of a regular surface M. Let $\mathbf{n} = \mathbf{X}_u \times \mathbf{X}_v/|\mathbf{X}_u \times \mathbf{X}_v|$. Then*

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 2\lambda^2 H\mathbf{n}$$

where H is the mean curvature, i.e. $H = \frac{1}{2}\frac{eG-2fG+gE}{EG-F^2}$, where e, f, g are the coefficients of the second fundamental form.

Proof. (Sketch)

$$\langle \mathbf{X}_{uu}+\mathbf{X}_{vv}, \mathbf{X}_u\rangle = \frac{1}{2}\langle \mathbf{X}_u, \mathbf{X}_u\rangle_u - \langle \mathbf{X}_v, \mathbf{X}_{uv}\rangle = \frac{1}{2}\langle \mathbf{X}_u, \mathbf{X}_u\rangle_u - \frac{1}{2}\langle \mathbf{X}_v, \mathbf{X}_v\rangle_v = 0.$$

Similarly, $\langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{X}_v\rangle = 0$. Hence

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = \langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{n}\rangle \mathbf{n} = (e+g)\mathbf{n} = 2\lambda^2 H\mathbf{n}$$

because

$$H = \frac{1}{2}\frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2}\frac{e+g}{\lambda^2}.$$

□

Corollary 1. *Suppose* $\mathbf{X}(u,v)$ *is an an isothermal coordinate parametrization of a regular surface* M*.* M *is a minimal surface if and only if* $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$*. (That is: each coordinate function is harmonic as a function of* u, v*.)*

Remark 1. Let $\mathbf{X}(u,v)$ be a coordinate parametrization of M. Let $\phi_1 = x_u - \sqrt{-1}x_v$, $\phi_2 = y_u - \sqrt{-1}y_v$, $\phi_3 = z_u - \sqrt{-1}z_v$. Then

(i) $\mathbf{X}$ is isothermal if and only if $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$.

(ii) M is minimal if and only if ϕ_i are analytic for $i = 1, 2, 3$.

First variational formula for area

Let $\mathbf{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a coordinate parametrization of a regular surface M. Let $\overline{D}$ be a compact domain in U and let $Q = \mathbf{X}(D) \subset M$. Let $h(u,v)$ be a smooth function on $\overline{D}$. Let $\mathbf{n} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ be the unit normal of the surface. Define:

$$\mathbf{Y}(u,v;t) = \mathbf{X}(u,v) + th(u,v)\mathbf{n}(u,v).$$

Lemma 2. *There exists* $\epsilon > 0$ *such that for each fixed* t *with* $|t| < \epsilon$*,* $\mathbf{Y}(u,v;t)$ *represent a parametrized regular surface. (*$\mathbf{Y}(u,v;t)$ *is called a* **normal variation** *of* $\overline{Q}$*.)*

Proof. (Sketch) $\mathbf{Y}_u = \mathbf{X}_u + t(h_u\mathbf{n} + h\mathbf{n}_u)$, etc. So

$$\begin{aligned}\mathbf{Y}_u \times \mathbf{Y}_v =& \mathbf{X}_u \times \mathbf{X}_v + t\left[(h_u\mathbf{n} + h\mathbf{n}_u) \times \mathbf{X}_v + \mathbf{X}_u \times (h_v\mathbf{n} + h\mathbf{n}_v)\right] \\ &+ t^2(h_u\mathbf{n} + h\mathbf{n}_u) \times (h_u\mathbf{n} + h\mathbf{n}_u) \\ =& \mathbf{X}_u \times \mathbf{X}_v + R(u,v,t).\end{aligned}$$

Since $|\mathbf{X}_u \times \mathbf{X}_v| \geq C_1$ for some $C_1 > 0$ on $\overline{D}$ and $|R| \leq \epsilon C_2$ for some $C_2 > 0$ on $\overline{D}$ independent of ϵ. So $\mathbf{Y}_u \times \mathbf{Y}_v \neq \mathbf{0}$ if ϵ is small enough. □

Let $\epsilon > 0$ be as above. Define $A(t)$ to be the area of

$$M(t) = \{\mathbf{Y}(u,v,t) | (u,v) \in \overline{D}\}.$$

Theorem 1 (First variation of area)**.**

$$\frac{dA}{dt}\Big|_{t=0} = -2\iint_{\overline{Q}} hH dA$$

where H is the mean curvature of M. Here for any function ϕ on $\overline{D}$,

$$\iint_{\overline{Q}} \phi dA := \iint_{\overline{D}} \phi|\mathbf{X}_u \times \mathbf{X}_v| dudv.$$

Proof. (Sketch) Let $E(u,v,t) = \langle \mathbf{Y}_u(u,v,t), \mathbf{Y}_u(u,v,t)\rangle$ etc. Let $E_0(u,v) = E(u,v,0)$ etc (which are the coefficients of the first fundamental form of $\mathbf{X}$).

$$\begin{aligned}
E(u,v,t) =& E_0(u,v) + 2th(u,v)\langle \mathbf{n}_u, \mathbf{X}_u\rangle + O(t^2) = E_0(u,v) - 2th(u,v)e(u,v) + O(t^2);\\
F(u,v,t) =& F_0(u,v) + 2th(u,v)\langle \mathbf{n}_u, \mathbf{X}_v\rangle + O(t^2) = F_0(u,v) - 2th(u,v)f(u,v) + O(t^2);\\
G(u,v,t) =& G_0(u,v) + 2th(u,v)\langle \mathbf{n}_v, \mathbf{X}_v\rangle + O(t^2) = G_0(u,v) - 2th(u,v)g(u,v) + O(t^2),
\end{aligned}$$

where e, f, g are the coefficients of the second fundamental form of $\mathbf{X}$. Hence

$$EG - F^2 = E_0G_0 - F_0^2 - 2t\left(eG_0 - 2fF_0 + gG_0\right) + O(t^2).$$

Hence

$$\begin{aligned}
A(t) =& \iint_{\overline{D}} \sqrt{(EG - F^2)} dudv\\
=& \iint_{\overline{D}} \sqrt{E_0G_0 - F_0^2} dudv - t\iint_{\overline{D}} h\frac{eG_0 - 2fF_0 + gG_0}{\sqrt{E_0G_0 - F_0^2}} dudv + O(t^2)\\
=& \iint_{\overline{D}} \sqrt{E_0G_0 - F_0^2} dudv - 2t\iint_{\overline{Q}} hHdA + O(t^2).
\end{aligned}$$

□

Corollary 2. *$A'(0) = 0$ for all normal variation of $\overline{Q}$ if and only if $H \equiv 0$ on Q. Actually, a regular surface M is minimal if and only if $A'(0) = 0$ for all normal variation of M with compact support: i.e. any variation by $f\mathbf{n}$ where f has satisfies $\overline{f \neq 0}$ is a compact set in M.*

To prove the theorem, we need to construct a so-called *bump function*, starting with

$$\phi(t) = \begin{cases} 0, & \text{if } t \le 0;\\ e^{-\frac{1}{t}}, & \text{if } t > 0.\end{cases}$$

Consider the function:

$$\Phi(t) = \frac{\psi_1(t)}{\psi_1(t) + \psi_2(t)}$$

where

$$\psi_1(t) = \phi(2+t)\phi(2-t), \psi_2(t) = \phi(t-1) + \phi(-1-t).$$

Then $\Phi(t)$ satisfies $\Phi(t) \geq 0$, and

$$\Phi(t) = \begin{cases} 1, & \text{if } |t| \leq 1; \\ 0, & \text{if } |t| \geq 2. \end{cases}$$

A reference for minimal surfaces: *Osserman, A survey of minimal surfaces.*

Assignment 10, Due Friday Nov 28, 2014

(1) Show that the helicoid:

$$\mathbf{X}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au),$$

and the Enneper's surface

$$\mathbf{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

are minimal surfaces.

(2) Let M be the surface of revolution by rotating the $(\phi(v), 0, v)$ about z-axis, so that M is parametrized by $\mathbf{X}(u,v) = (\phi(v)\cos u, \phi(v)\sin u, v)$ Show that the mean curvature (w.r.t. to the unit normal $\frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$) of the surface is given by

$$H = -\frac{1 + (\phi')^2 - \phi\phi''}{2\phi\left(1 + (\phi')^2\right)^{\frac{3}{2}}}.$$

(3) Let $\mathbf{X}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ be a local parametrization of a regular surface M such that the coefficients of the first fundamental form satisfy $E = G$ and $F = 0$. Let $\xi_i = \frac{\partial x_i}{\partial u}$, $\eta_i = \frac{\partial x_i}{\partial v}$. Prove that if M is minimal then for each i, the functions ξ_i, η_i satisfy the Cauchy-Riemann equations:

$$\frac{\partial \xi_i}{\partial u} = -\frac{\partial \eta_i}{\partial v}, \quad \frac{\partial \xi_i}{\partial v} = \frac{\partial \eta_i}{\partial u}.$$

(Hence $\xi_i - \sqrt{-1}\eta_i$ is analytic.)

Isaac Newton: *"I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."*