## 1. Some remarks

(i) Geodesic curvature is intrinsic in the following sense. Let $M_1, M_2$ be regular oriented surfaces and $F : M_1 \to M_2$ be an isometry and is orientation preserving: i.e. if $\mathbf{v}, \mathbf{w}$ are positively oriented in $M_1$, then $dF(\mathbf{v}), dF(\mathbf{w})$ are positively oriented on $M_2$. Suppose $\alpha(s)$ is a regular curve on $M_1$. Let $\beta(s) = F \circ \alpha(s)$. Then the geodesic curvature of $\alpha$ at $\alpha(s)$ is equal to the geodesic curvature of $\beta$ at $\beta(s)$.

(ii) Let $M_1, M_2$ be regular surfaces. A smooth map $F : M_1 \to M_2$ is a *local isometry* if for any $p \in M_1$ there is neighborhood $U$ of $p$ and a neighborhood $V$ of $q = F(p)$ such that $F : U \to V$ is an isometry.

(iii) A local isometry will map geodesics to geodesics.

(iv) Let $F$ be a Euclidean motion on $\mathbb{R}^3$, i.e. $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b_0}$ where $A$ is a orthogonal matrix and $\mathbf{b_0}$. Let $M$ be a regular surface and let $N = F(M)$. Then $F : M \to N$ is an isometry.

## 2. Geodesics of surfaces of revolution

Let $(\phi(v), 0, \psi(v))$ be a regular curve on the $xz$-plane such that:

(i) $\phi(v) > 0$, i.e. the curve does not intersect the $z$-axis.

(ii) $(\phi_v)^2 + (\psi_v)^2 = 1$, i.e. the curve is parametrized by arc length.

Consider the surface of revolution $M$ given by

$$\mathbf{X}(u, v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)).$$

The curve $\mathbf{X}(u_0, v)$ where $u_0$ is a constant is called a *meridian*; and the curve $\mathbf{X}(u, v_0)$ where $v_0$ is a constant is called a *parallel*.

**Lemma 1.** *The first fundamental form is given by:*

$$\begin{cases} g_{11} = & E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = \phi^2, ; \\ g_{12} = & g_{21} = F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 \\ g_{22} = & G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = (\phi_v)^2 + (\psi_v)^2 = 1. \end{cases}$$

*The Christoffel symbols are:* $\Gamma_{12}^1 = \Gamma_{21}^1 = \phi_v/\phi$, $\Gamma_{11}^2 = -\phi\phi_v$ *and all other* $\Gamma_{ij}^k$ *are zeros.*

*Proof.* (Sketch) $g^{11} = 1/\phi^2$, $g^{12} = g^{21} = 0$, $g^{22} = 1$. Hence

$$\Gamma_{ij}^1 = \frac{1}{2} g^{1m} \left( g_{im,j} + g_{mj,i} - g_{ij,m} \right) = \frac{1}{2\phi^2} \left( g_{i1,j} + g_{1j,i} - g_{ij,1} \right).$$

Hence

$$\begin{cases} \Gamma_{11}^1 = & \frac{1}{2\phi^2} g_{11,u} = 0; \\ \Gamma_{12}^1 = & \Gamma_{21}^1 = \frac{1}{2\phi^2} \left( g_{11,v} + g_{12,u} - g_{12,u} \right) = \frac{\phi_v}{\phi}; \\ \Gamma_{22}^1 = & -\frac{1}{2\phi^2} g_{22,u} = 0 \end{cases}$$

$$\Gamma^2_{ij} = \frac{1}{2} g^{2m} \left( g_{im,j} + g_{mj,i} - g_{ij,m} \right) = \frac{1}{2} \left( g_{i2,j} + g_{2j,i} - g_{ij,2} \right).$$

$$\begin{cases} \Gamma^2_{11} = & -\frac{1}{2} g_{11,2} = -\phi\phi_v; \\ \Gamma^2_{12} = & \Gamma^2_{21} = 0 \\ \Gamma^2_{22} = & -\frac{1}{2} g_{22,v} = 0 \end{cases}$$

□

**Lemma 2.** *$\alpha(t) = \mathbf{X}(u(t), v(t))$ is a geodesic if and only if*

$$\begin{cases} u'' + & \frac{2\phi_v}{\phi} u'v' = 0, \\ v'' - & \phi\phi_v (u')^2 = 0. \end{cases}$$

**Corollary 1.** *Any meridian is a geodesic. A parallel $\mathbf{X}(u, v_0)$ is a geodesic if and only if $\phi_v(v_0) = 0$.*

To study the behavior of general geodesics, first we have the following lemma:

**Lemma 3.** *Let $a_1(t), a_2(t)$ be smooth functions on $(T_1, T_2) \subset \mathbb{R}$ such that $a_1^2 + a_2^2 = 1$. For any $t_0 \in (T_1, T_2)$ and $\theta_0$ such that $a_1(t_0) = \cos\theta_0$, $a_2(t_0) = \sin\theta_0$, there exists unique a smooth function $\theta(t)$ with $\theta(t_0) = \theta_0$ such that $a_1(t) = \cos\theta(t)$ and $a_2(t) = \sin\theta(t)$.*

*Proof.* (Sketch) Suppose $\theta$ satisfies the condition. Then $a_1' = -\theta' \sin\theta$, $a_2' = \theta' \cos\theta$. Hence $\theta' = a_1 a_2' - a_2 a_1'$. From this we have uniqueness.

To prove existence, fix $t_0 \in (T_1, T_2)$ and let $\theta_0$ be such that $\cos\theta_0 = a_1(0)$, $\sin\theta_0 = a_2(0)$. Let

$$\theta(t) = \theta_0 + \int_{t_0}^{t} (a_1 a_2' - a_2 a_1') d\tau$$

and let $b_1 = \cos\theta(t)$, $b_2 = \sin\theta(t)$. Let $f = (a_1 - b_1)^2 + (a_2 - b_2)^2$, then one can check that $f' = 0$. So $\theta$ is a smooth function and is a required function. □

Now let $\alpha(s) = \mathbf{X}(u(s), v(s))$ be a geodesic on $M$ parametrized by arc length. Let $\mathbf{e_1} = \mathbf{X}_u / |\mathbf{X}_u|$ and $\mathbf{e_2} = \mathbf{X}_v / |\mathbf{X}_v|$.

$$\alpha' = a_1 \mathbf{e_1} + a_2 \mathbf{e_2}.$$

By the lemma there exists smooth function $\theta(s)$ such that $a_1 = \sin\theta$, $a_2 = \theta$. Note that $\theta$ is the angle between $\alpha'$ and the meridian.

**Proposition 1** (Clairaut's Theorem)**.** *$r(s)\sin\theta(s)$ is constant along $\alpha$, where $r(s)$ is the distance of $\alpha(s)$ from the $z$-axis.*

*Proof.* (Sketch) Denote the $\frac{d\alpha}{ds}$ by $\alpha'$ etc. Since $r(s) = \phi(v(s))$,

$$r' = \phi_v v'.$$

Also $\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = u'\phi$, so $(\sin\theta)' = u''\phi + u'v'\phi_v$.

$$\begin{aligned}(r\sin\theta)' =& \phi_v v' u' \phi + u''\phi + \phi_v u' v' \\ =& \phi\left(u'' + \frac{2\phi_v}{\phi}u'v'\right) \\ =& 0.\end{aligned}$$

□

Let us analyse a geodesic $\alpha(s)$, $0 \le s < L \le \infty$, on the surface of revolution parametrized by arc length. Assume $\alpha$ is not a parallel. Let us also assume that $\psi(v)$ is increasing. Let $r(s)$ and $\theta(s)$ as the theorem. Let $\theta_0 = \theta(0)$. We may assume that $0 \le \theta_0 \le \frac{\pi}{2}$. By the theorem, $r(s)\sin\theta(s) = R$ for some constant $R \ge 0$. Note that $r(s) \ge R$.

- If $R = 0$, then $\alpha$ is a meridian.
- $R > 0$. Note $\cos^2\theta = 1 - \frac{R^2}{r^2}$. Then the geodesic will go up for all $s$, as long as $r > R$, i.e. the $z$ coordinate of $\alpha$ is increasing in $s$. Either $\alpha$ does not come close to any parallel of radius $R$, and $\alpha$ will go up for all $s$, or $\alpha$ will be close to a parallel $C$ of radius $R$. Let $C$ be the first such parallel above $\alpha$. Then we have the following cases:
  (i) $C$ is a geodesic. Then $\alpha$ will not meet $C$. (Why?) So $\alpha$ must come arbitrarily close to $C$ without intersecting $C$.
  (ii) $\alpha(s_0) \in C$ for some $s_0$. At such a point $\theta(s_0) = \frac{\pi}{2}$. $C$ is not a geodesic, so at this point $\phi_v \ne 0$, and so $\phi_v < 0$. (Why?) Hence the parallels just above $C$ should have smaller radius. Hence $\alpha$ must bounce off from $C$ and turn downward.

**Assignment 9, Due Friday Nov 21, 2014**

(1) Write down the differential equations for the geodesics on the torus:

$$\mathbf{X}(u, v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$$

with $a > r > 0$. Also, show that if $\alpha$ is a geodesic start at a point on the topmost parallel $(a\cos u, a\sin u, r)$ and is tangent to this parallel, then $\alpha$ will stay in the region with $-\pi/2 \le v \le \pi/2$.

(2) Let $\alpha$ be a geodesic on a surface of revolution. Using the same notation as in the Clairuat's Theorem, suppose $r(s)\sin\theta(s) = R$

which is a positive constant. Prove that if $\alpha$ is not a parallel, then $\alpha$ will not intersect any parallel which is a geodesic and has radius $R$.

(3) Let $\mathbf{X} : U \to M$, $(u_1, u_2) \to \mathbf{X}(u_1, u_2)$, be a coordinate parametization, with $U$ being an open set in $\mathbb{R}^2$. Suppose the first fundamental form in this coordinate satisfies $g_{12} = 0$, and $g_{11} = g_{22} = \exp(2f)$ for some smooth function $f$, i.e. $g_{ij} = \exp(2f)\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and is zero if $i \neq j$. Show that the Christoffel symbols are

$$\Gamma_{ij}^k = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k$$

where $f_i = \frac{\partial f}{\partial u_i}$ etc.

(4) With the same assumptions and notation as in the previous exercise. $\mathbf{e_1} = \mathbf{X}_1/|\mathbf{X}_1|$, $\mathbf{e_2} = \mathbf{X}_2/|\mathbf{X}_2|$, and $\mathbf{n} = \mathbf{e_1} \times \mathbf{e_2}$. Let $\alpha(s)$ be a geodesic on $M$ such that $\alpha(s) = \mathbf{X}(u_1(s), u_2(s))$. Let $\theta(s)$ be a smooth function on $s$ such that $\alpha'(s) = \mathbf{e_1}(s)\cos\theta(s) + \mathbf{e_2}(s)\sin\theta(s)$, where $\mathbf{e_i}(s) = \mathbf{e_i}(\alpha(s))$. Show that $\mathbf{a} := \mathbf{n} \times \alpha' = -\mathbf{e_1}(s)\sin\theta(s) + \mathbf{e_2}(s)\cos\theta(s)$. Show also that

$$\begin{aligned} k_g &= -\langle \alpha', \mathbf{a}' \rangle \\ &= \exp(-2f)\langle \frac{d}{ds}\mathbf{X}_1, \mathbf{X}_2 \rangle + \theta' \\ &= \left( -u' \frac{\partial f}{\partial v} + v' \frac{\partial f}{\partial u} \right) + \theta'. \end{aligned}$$

(Note that if $f = 1$, i.e. $M$ is a plane, then $k_g = \theta'$.)