MAT 4030: DIFFERENTIAL GEOMETRY I
2014-15, 1st term

Instructor: Tam Luen Fai, Rm 704, AB1, tel: 39438066; email: lftam@math.cuhk.edu.hk

TA: Huang Shao Chuang, LSB222A, tel: 39433575;
email: schuang@math.cuhk.edu.hk
Office hour: to be announced;

References:

Oprea: *Differential geometry and its applications* ;

Do Carmo: *Differential geometry of curves and surfaces*;

Klingenberg: *A course in Differential Geometry*;

Spivak: *A comprehensive introduction to Differential Geometry, Vol. 2*

Hilbert and Cohbn-Vossen: *Geometry and the imagination.*

(We will basically follow the book by Oprea.)

Assessment Scheme: Homework 10%; Midterm 30%, Final Exam 60%.

Assignment 1, Due 12/9/2014

(1) (The tractrix) Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right).$$

(a) Prove that α is regular except at $t = \frac{\pi}{2}$.

(b) Prove that the length of the segment of the tangent of α between the point of tangency and the y-axis is constantly 1.

(2) Let $\alpha(t)$ be a regular parametrized curve (not necessary by arc length). Show that the curvature and torsion are given by

$$k(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}, \quad \tau(t) = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}.$$

Also compute the curvature and torsion of the circular helix.

(3) Assume that $k(s) > 0$, $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parametrized by arc length. Show that α lies on a sphere if and only if

$$\rho^2 + (\rho')^2 \lambda^2 = \text{constant}.$$

where $\rho = 1/k(s)$, $\lambda = 1/\tau$.

(*Hint*: Necessity: Differentiate $|\alpha|^2$ three times to obtain $\alpha = -\rho N - \rho' \lambda B$. Sufficiency: Show that $\beta = \alpha + \rho N - \rho' \Gamma B$ is constant.)

The Frenet formula

Let $\alpha(s)$ be the regular curve parametrized by arc length.
Let $\vec{T} = \alpha'$. Then

$$\begin{aligned} k(s) :=& |T'|(s) \quad \text{(curvature)};\\ N(s) :=& \frac{1}{k(s)} T'(s) \quad \text{(normal, if } k > 0);\\ B(s) :=& T(s) \times N(s) \quad \text{(binormal, if } k > 0). \end{aligned}$$

Fact: $B' = -\tau N$, τ is called the torsion of α.

Proposition 1. *$\alpha(s)$ be the regular curve parametrized by arc length.*

(i) $k(s) \equiv 0$ *if and only if α is a straight line.*
(ii) $\tau \equiv 0$ *if and only if α is a straight line.*
(iii) *α is a plane curve with $k(s) = k_0 > 0$, then α is a circular arc.*

Theorem 1. *(Frenet formula) Let α be a regular curve with curvature $k > 0$. Then*

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Some results of the local theory of curves

Theorem 2. *Let $k(s) > 0$ and $\tau(s)$ be smooth function on (a, b). There exists a regular curve $\alpha : (a, b) \to \mathbb{R}^3$ with $|\alpha'| = 1$, such that the curvature and torsion of α are k, τ respectively.*

Moreover, α is unique in the sense that if β is another curve satisfying the above conditions, then $\beta(s) = A\alpha(s) + \vec{c}$ for some constant orthogonal matrix A and some constant vector $\vec{c}$.

Theorem 3. *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc-length so that $k > 0$. Assume $0 \in I$ and $T(0) = e_1, N = e_2, B = e_3$, where $\{e_i\}$ is the standard ordered basis for $\mathbb{R}^3$. Then for $|s|$ small, $\alpha(s) = (x(s), y(s), z(s))$ is given by*

$$\begin{cases} x(s) = s - \frac{1}{6}k^2 s^3 + O(|s|^4) \\ y(s) = \frac{1}{2}ks^2 + \frac{1}{6}k's^3 + O(|s|^4) \\ z(s) = \frac{1}{6}k\tau s^3 + O(|s|^4). \end{cases}$$