Suggested Solutions of Assignment 9

(1) Geodesics equations for torus:

$$\begin{cases} u'' - 2\dfrac{r\sin v}{(a+r\cos v)} u'v' = 0 \\ v'' + \dfrac{(a+r\cos v)}{r}(\sin v)\, u'^2 = 0 \end{cases}.$$

By Prop 1 in Hand-out 10 (Clairaut's Thm),

$(a+r\cos v(s))\sin\phi(s) = c$ where c is a constant and ϕ is the angle between α' and the meridian X_v.

~~If α is a geodesic Since α starts at $(a\cos u, a\sin u, r)$~~

~~and~~ Since α starts at a point on the top-most parallel $(a\cos u, a\sin u, r)$ and is tangent to this parallel, we have $\phi = \frac{\pi}{2}$ at that pt, then we get $a = c$. But

$$a + r\cos v(s) = \frac{c}{\sin\phi(s)} \geq a \quad (\text{note that } \sin\phi(s) > 0.)$$

$\Rightarrow \quad \cos v(s) \geq 0$

$\Rightarrow \quad \alpha$ will stay in the region with $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

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(2) Pf: We argue by contradiction. Assume α intersects ~~a~~ a parallel which is geodesic and has radius R at p, then by Clairaut's Thm $\phi = \frac{\pi}{2}$ at p. It means $\dot{\alpha}$ is tangent to this parallel. By the existence and uniqueness of the geodesics equations, we see that α is part of this parallel which contradicts with our assumption.

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(3)

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

$$= \frac{1}{2} e^{-2f} \delta_{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

$$= \frac{1}{2} e^{-2f} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right)$$

$$= \frac{1}{2} e^{-2f} \left(2e^{2f} \cdot f_i \delta_{jk} + 2e^{2f} \cdot f_j \delta_{ik} - 2e^{2f} \cdot f_k \delta_{ij} \right)$$

$$= \delta_{jk} f_i + \delta_{ik} f_j - \delta_{ij} f_k .$$

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(4) $a \triangleq n \times \alpha' = (e_1 \times e_2) \times (e_1 \cos\theta + e_2 \sin\theta)$

$$= e_2 \cos\theta - e_1 \sin\theta .$$

We first compute $|\alpha'| = 1$, then we see α is parametrized by arc-length.

$$K_g \triangleq \langle \alpha'', n\times\alpha' \rangle$$

$$= \underbrace{\langle \alpha', n\times\alpha' \rangle'}_{0} - \langle \alpha', a' \rangle$$

$$= -\langle \alpha', a' \rangle$$

$$= -\langle e_1\cos\theta + e_2\sin\theta , (-e_1\sin\theta + e_2\cos\theta)' \rangle$$

$$= -\langle e_1\cos\theta + e_2\sin\theta , -e_1'\sin\theta - e_1\cos\theta\,\theta' + e_2'\cos\theta - e_2\sin\theta\cdot\theta$$

$$= -\langle e_1\cos\theta , -e_1\cos\theta\cdot\theta' + e_2'\cos\theta \rangle$$

$$-\langle e_2\sin\theta , -e_1'\sin\theta - e_2\sin\theta\cdot\theta' \rangle$$

$$\left(\begin{array}{l} \text{note that} \\ \langle e_1, e_2\rangle = 0 \\ \langle e_1, e_1'\rangle = 0 \\ \langle e_2, e_2'\rangle = 0 \end{array} \right)$$

$$= \theta' + \langle e_1', e_2 \rangle$$

$$= \theta' + e^{-2f}\langle X_1', X_2 \rangle$$

$$\left(\begin{array}{l} X_1 = e^f e_1 \\ X_1' = e^f f' e_1 + e^f e_1' \\ X_2 = e^f e_2 \\ \langle X_1', X_2\rangle = e^{2f}\langle e_1', e_2\rangle \end{array} \right)$$

$$= \theta' + e^{-2f}\langle X_{11}u' + X_{12}v', X_2 \rangle$$

$$= \theta' + (-u'\cdot f_v + v'\cdot f_u).$$

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$$\left(\begin{array}{l} \langle X_{12}, X_2\rangle \\ = \frac{1}{2}\langle X_2, X_2\rangle_1 \\ = \frac{1}{2} e^{2f}\cdot 2f_u \\ = e^{2f} f \end{array} \right)$$

$$\left(\begin{array}{l} \langle X_{11}, X_2\rangle = -\langle X_1, X_{12}\rangle \\ = -\frac{1}{2}\langle X_1, X_1\rangle_2 \\ = -\frac{1}{2} e^{2f} 2\cdot f_v \\ = -e^{2f}\cdot f_v \end{array} \right)$$