Suggested Solutions of Assignment 8

(1) (a) We may parametrize the ellipse in the following way:

$$\alpha(t) = (a\cos t, b\sin t, 0), \quad t \in [0, 2\pi).$$

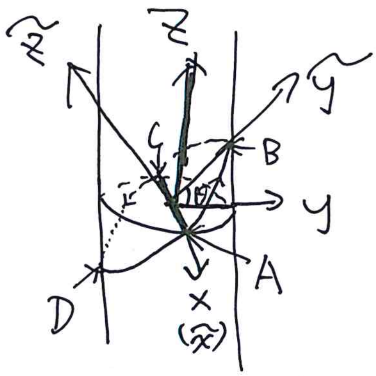
$$\alpha' = (-a\sin t, b\cos t, 0), \quad \alpha'' = (-a\cos t, -b\sin t, 0)$$

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{|(0,0,ab)|}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

$$k((a,0)) = \frac{a}{b^2}, \quad k((0,b)) = \frac{b}{a^2}.$$

(b) We first show α is an ellipse. We see that α is in the ~~plane~~ $\tilde{x}\tilde{y}$-plane with normal "$\tilde{z}$" where

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$



Points in α satisfy
$$\begin{cases} x^2 + y^2 = 1 \\ z = y\tan\theta \end{cases}$$

$$\Rightarrow \quad \tilde{x}^2 + \frac{\tilde{y}^2}{\left(\frac{1}{\cos^2\theta}\right)} = 1, \quad \tilde{z} = 0$$

So α is an ellipse in $\tilde{x}\tilde{y}$-plane.

Next, we compute the absolute value of the geodesic curvature of α at A, B, C, D (See the picture above).

We want to apply $K_g = K_\alpha \cdot \cos\varphi$ where φ = angle between Π (normal) and $B(T\times N)$. Now $a=1$, $b=\frac{1}{\cos\theta}$

$$|K_g(A)| = \cos^2\theta \cdot \cos\frac{\pi}{2} = 0 \left(= |K_g(C)|\right)$$

$$|K_g(B)| = \frac{1}{\cos\theta} \cdot \sin\theta = \tan\theta \left(= |K_g(D)|\right)$$

(Rmk: Assume $\theta \in [0, \frac{\pi}{2})$.)

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(2) $(\Leftarrow)$ We assume $(\alpha'')^T = \lambda(\tau)\alpha'$. Consider a new parametrization $t = \int_a^\tau \exp\left(\int_a^\rho \lambda(s)ds\right) d\rho$. Denote $\dot{\alpha} = \frac{\partial\alpha}{\partial t}$, etc.

Note that $(\ddot{\alpha})^T = \left(\lambda(\tau)\left(\frac{d\tau}{dt}\right)^2 + \frac{d^2\tau}{dt^2}\right)\alpha'$ and

$$\frac{d\tau}{dt} = e^{-\int_0^\tau \lambda(s)ds}, \quad \frac{d^2\tau}{dt^2} = e^{-2\int_a^\tau \lambda(s)ds} \cdot (-\lambda(\tau)).$$

Thus $(\ddot{\alpha})^T = 0$ i.e α is a geodesic after reparametrizat

$(\Rightarrow)$ We assume α is a geodesic after reparametrization.

i.e $(\ddot{\alpha})^T = 0$, $t = t(\tau)$.

i.e $(\alpha'')^T \left(\frac{d\tau}{dt}\right)^2 + \alpha' \frac{d^2\tau}{dt^2} = 0$

So $(\alpha'')^T = -\frac{d^2\tau}{dt^2} \Big/ \left(\frac{d\tau}{dt}\right)^2 \cdot \alpha'$. Let $\lambda(\tau) = \frac{-\ddot{\tau}}{(\dot{\tau})^2}$. #

(3) See Tutorial 9.