Suggested Solutions of Assignment 10:

(1) Helicoid: $X_u = (-a\sinh v \sin u,\ a\sinh v\cos u,\ a)$

$$X_v = (a\cosh v\cos u,\ a\cosh v \sin u,\ 0)$$

$$\Rightarrow \quad \langle X_u, X_v\rangle = 0, \quad |X_u| = |X_v| \quad (\text{Isothermal})$$

By Prop. 1 in handout-11, to see $X$ is a minimal surf
we only need to show $X_{uu} + X_{vv} = \vec{0}$. $(*)$

$$X_{uu} = (-a\sinh v\cos u,\ -a\sinh v\sin u,\ 0)$$

$$X_{vv} = (a\sinh v\cos u,\ a\sinh v \sin u,\ 0)$$

$$\Rightarrow (*).$$

Enneper's surface:

$$X_u = (1-u^2+v^2,\ 2uv,\ 2u)$$

$$X_v = (2uv,\ 1-v^2+u^2,\ -2v)$$

$$\Rightarrow \langle X_u, X_v\rangle = 0, \quad |X_u| = |X_v| \quad (\text{Isothermal})$$

$$X_{uu} = (-2u,\ 2v,\ 2), \quad X_{vv} = (2u,\ -2v,\ -2)$$

$$\Rightarrow X_{uu} + X_{vv} = 0 \Rightarrow X \text{ is a minimal surface.}$$

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(2) $X(u,v) = (\phi(v)\cos u, \phi(v)\sin u, v)$

$X_u = (-\phi\sin u, \phi\cos u, 0)$, $X_v = (\phi'\cos u, \phi'\sin u, 1)$

$X_{uu} = (-\phi\cos u, -\phi\sin u, 0)$, $X_{uv} = (-\phi'\sin u, \phi'\cos u, 0)$

$X_{vv} = (\phi''\cos u, \phi''\sin u, 0)$, $U \doteq \dfrac{X_u \times X_v}{|X_u \times X_v|} = \dfrac{(\cos u, \sin u, -\phi')}{\sqrt{1+\phi'^2}}$

$E = \phi^2$, $F = 0$, $G = \phi'^2 + 1$

$e = \dfrac{-\phi}{\sqrt{1+\phi'^2}}$, $f = 0$, $g = \dfrac{\phi''}{\sqrt{1+\phi'^2}}$

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$$H = \frac{1}{2}\frac{eG - 2fG + gE}{EG - F^2} = \frac{1}{2}\cdot\frac{-\phi(\phi'^2+1)^{\frac{1}{2}} + \phi^2\phi''\frac{1}{\sqrt{1+\phi'^2}}}{\phi^2\cdot(\phi'^2+1)}$$

$$= -\frac{1+(\phi')^2 - \phi\phi''}{2\phi(1+(\phi')^2)^{\frac{3}{2}}}.$$

(Note that $\phi \neq 0$ otherwise $(\phi(v), 0, v)$ touch $z$-axis.)

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(3) $M$ is minimal $\xRightarrow{\text{by Prop 1 in handout 11}}$ $X_{uu} + X_{vv} = 0$.

$\Rightarrow X^i_{uu} + X^i_{vv} = 0 \quad \forall i = 1,2,3.$

$\Rightarrow \dfrac{\partial \xi_i}{\partial u} = -\dfrac{\partial \eta_i}{\partial v}$ 2

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("$\dfrac{\partial \xi_i}{\partial v} = \dfrac{\partial \eta_i}{\partial u}$" follows from the fact $x_i$ is $C^\infty$ $\forall i=1,2,3$