THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Spring MATH2230
Tutorial 3

0.1 Logarithmic function

If the logarithmic is the "inverse" of exponential function, we let $z_0 = re^{i\theta} \neq 0$ for $-\pi < \theta \leq \pi$ ($\theta = Arg(z)$) and we expect to have

$$\log(z_0) = \log(re^{i\theta}) = \log(r) + i\theta$$

However, $z_0 = e^{i\theta} = e^{i\theta + 2k\pi i}$ for any integers, hence we have

$$\log(z_0) = \log(re^{i\theta + 2k\pi i}) = \log(r) + i\theta + 2k\pi i$$

It shows that the logarithmic defined in this way is not a function, we see that $\log(z_0)$ represent a set $\log(z_0) := \{\log(r) + i\theta + 2k\pi i | k \in \mathbb{Z}\}$ or we will call log is a multiple-valued function.

Definition 1. *The principal value of* $\log(z)$ *equals to*

$$Log(z) = \log|z| + iArg(z)$$

where $-\pi < Arg(z) \leq \pi$.

Remark 1 : Since it is reasonable to define the range of the angle of z_0 in another way, say, $z_0 = re^{i\theta} \neq 0$ for $a < \theta \leq 2\pi + a$ for any real number a. Such a choice of range of the angle of z is called branch. And we can define another single-value function for log by $\log(r) + i\theta$ with $a < \theta \leq 2\pi + a$. (The word "principal" in definition 1 means that $a = -\pi$. We would not call the single-value log to be principal if $a \neq 0$.) And the range $-\pi < \theta \leq \pi$ is called principal branch.

Remark 2 : Although $Log(z)$ can be defined on the ray $\theta = a$, $Log(z)$ is not continuous there (not analytic).

Remark 3 : $Log(z)$ is analytic in the domain $r > 0$ and $-\pi < Arg(z) < \pi$ (or other branch).

0.2 Power function

Definition 2. *Let* $z \neq 0$ *and* $c \in \mathbb{C}$, *the power is defined as*

$$z^c = e^{c\ Log(z)}$$

Clearly it can be defined for other branch.

0.3 Trigonometric function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = -i\sin(iz) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \cos(iz) = \frac{e^{z} + e^{-z}}{2}$$

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$
$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

0.4 Integraion

Definition 3. *Let $w(t) = u(t) + iv(t)$ be a complex function of a real variable t, the definite integral of $w(t)$ over the interval $a \le t \le b$ is defined as*

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Definition 4. *Let $z(t) = x(t) + iy(t) : [a, b] \to \mathbb{C}$ be a continuous complex function of a real variable t, $z(t)$ is a simple curve or Jordan curve if $z(t)$ is one to one (the curve does not intersect itself). It is closed if $z(a) = z(b)$. Such a curve is positive oriented when it is in the counterclockwise direction.*

Definition 5. *A contour is a piecewise smooth simple curve.*

Definition 6. *Let f be piecewise continuous on a contour C represented by $z(t) : [a, b] \to \mathbb{C}$. The line integral (contour integral) of f along C is defined to be*

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Definition 7.

$$\int_C f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

Proposition 1.

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz|$$

0.5 Exercise

1. Compute the value of $\log(-1 + \sqrt{3}i)$ with branch $-\pi < Arg(z) \le \pi$

2. Find the domain such that $f(z) = Log(z - i)$ is analytic.

3. Find the values of $(1 + i)^i$ and the principal value of it.

4. Use the Schwarz reflection principle to show that $\overline{\sin z} = \sin \overline{z}$ and $\overline{\cos z} = \cos \overline{z}$.

5. Compute the integral $\int_C f(z)dz$ with
(a) C is the arc of the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$ and $f(z) = \dfrac{z+2}{z}$
(b) C consists of the arc of the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$ and the line segment $z = x$ with $x \in [0, 2]$. $f(z) = z - 1$.