**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**2018 Spring MATH2230**
**Tutorial 10**

There are three types of isolated singularity. We suppose that $f$ is analytic function in $B_R(a) \setminus \{a\}$.(hence $a$ is isolated singularity)

**Definition 1.** *The point a is called a removable singularity if there is an analytic function* $\widetilde{f}$ *in* $B_R(a)$ *such that* $\widetilde{f} = f$ *in* $B_R(a) \setminus \{a\}$ *(*$\widetilde{f} = f$ *except at* $z = a$*)*

Remark : It is the best behaved singularity, it is 'almost' an analytic function.

**Theorem 1.** *The point a is a removable singularity iff* $\lim_{z\to a}(z-a)f(z) = 0$.

**Definition 2.** *The point a is called a pole if* $\lim_{z\to a} |f(z)| = \infty$.

**Theorem 2.** *If f has a pole at* $z = a$*, there is a positive integer m and an analytic function g in* $B_R(a)$ *such that* $f = \dfrac{g}{(z-a)^m}$.

**Definition 3.** *The point a is called an essential singularity if it is not neither removable singularity nor pole.*

Remark : In this definition, we can see that $\lim_{z\to a} |f(z)|$ fails to exist, it will converges to different finite value and $\infty$ according to different path taken.

**Theorem 3.** *(Casorati-Weierstrass theorem) If f has essential singularity at* $z = a$*, then for every* $\delta > 0$*, the closure of* $f(B_\delta(a) \setminus \{a\}) = \mathbb{C}$

Remark : It tells us that given any $c \in \mathbb{C}$, there is $z$ arbitrary close to $a$ such that $f(z)$ arbitrary close to $c$.

In the view of larrent series, we have the following conclusion,

**Theorem 4.** *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} \dfrac{b_n}{(z-a)^n}$ *be its Larrent series in* $B_R(a) \setminus \{a\}$*, then*

- $z = a$ *is a removable singularity iff* $b_n = 0$ *for* $n \geq 1$*,*
- $z = a$ *is a pole of order m iff* $b_m \neq 0$ *and* $b_n = 0$ *for* $n \geq m+1$
- $z = a$ *is an essential singularity iff* $b_n \neq 0$ *for infinitely many integers* $n \geq 1$*. (not necessary every n!)*

Exercise:

1. Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$.
2. Compute $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1}$.
3. Compute $\displaystyle\int_0^{\infty} \frac{x \sin 2x dx}{x^4+2}$.