# HW 8

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1. $f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z^2}\right)^{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}}$

2. $f(z) = \frac{1}{z} \cdot \frac{1}{1-(-\frac{1}{z})^n} = \frac{1}{z}\sum_{n=0}^{\infty} (-\frac{1}{z})^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$

5. In $D_1$, $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n$

In $D_2$, $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

$D_3$: $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$

6. $\frac{z}{(z-1)(z-3)} = \frac{z-3+3}{(z-1)(z-3)} = \frac{3}{(z-1)(z-3)} + \frac{1}{z-1}$

$\frac{1}{z-3} = \frac{1}{z-1-2} = \frac{-1}{2}\frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}$

$\therefore \frac{z}{(z-1)(z-3)} = \frac{3}{z-1}\left(-\frac{1}{2}\right)\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} + \frac{1}{z-1}$

$= -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{3}{2}\frac{1}{z-1} + \frac{1}{z-1} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$

7(a) $\frac{a}{z-a} = \frac{a}{z}\frac{1}{1-\frac{a}{z}} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$

(b) $\frac{a}{e^{i\theta}-a} = \frac{a}{\cos\theta - a + i\sin\theta} = \frac{a(\cos\theta - a - i\sin\theta)}{(\cos\theta-a)^2 + \sin^2\theta}$

$= \frac{a\cos\theta - a^2}{1-2a\cos\theta + a^2} - i\frac{a\sin\theta}{1-2a\cos\theta+a^2}$

$\sum_{n=1}^{\infty}\frac{a^n}{e^{i\theta n}} = \sum_{n=1}^{\infty} a^n(\cos n\theta - i\sin n\theta) = \sum_{n=1}^{\infty} a^n\cos n\theta - i\sum_{n=1}^{\infty} a^n\sin n\theta$

Compare the real parts and the imaginary parts, the result follows.

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2. $$\frac{1}{z^2} = \frac{1}{(1-(1-z))^2} = \frac{1}{(1-z)^2}\frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \frac{1}{(1-z)^2}\sum_{n=0}^{\infty}\frac{(n+1)}{(1-z)^n}$$
$$= \sum_{n=2}^{\infty}\frac{(n-1)(-1)^n}{(z-1)^n}$$

4. When $z \neq 0$
$$\frac{1}{z^2}\left(1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right) = \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}z^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2(n+1))!}z^{2n}$$
So $z=0$ is a removable singularity.
$$\lim_{z\to 0}\frac{1-\cos z}{z^2} = \lim_{z\to 0}\frac{\sin z}{2z} = \frac{1}{2}$$
So $f$ is entire.

5. Around $\frac{\pi}{2}$, $$\frac{\cos z}{z-\frac{\pi}{2}} = \frac{\cos\left(\frac{\pi}{2}+z-\frac{\pi}{2}\right)}{z-\frac{\pi}{2}} = \frac{-\sin\left(z-\frac{\pi}{2}\right)}{z-\frac{\pi}{2}}$$
$$= -\frac{1}{z-\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(z-\frac{\pi}{2}\right)^{2n+1}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\left(z-\frac{\pi}{2}\right)^{2n}$$
So $\frac{\cos z}{z-\frac{\pi}{2}}$ is analytic at $z=\frac{\pi}{2}$, which implies
$\frac{\cos z}{z^2-\left(\frac{\pi}{2}\right)^2} = \frac{1}{z+\frac{\pi}{2}}\frac{\cos z}{z-\frac{\pi}{2}}$ is analytic at $z=\frac{\pi}{2}$
By similar argument concerning $\frac{\cos z}{z+\frac{\pi}{2}}$,
$\frac{\cos z}{z^2-\left(\frac{\pi}{2}\right)^2}$ is analytic at $z=-\frac{\pi}{2}$
$$\lim_{z\to\frac{\pi}{2}}\frac{\cos z}{z^2-\left(\frac{\pi}{2}\right)^2} = \lim_{z\to\frac{\pi}{2}}\frac{-\sin z}{2z} = -\frac{1}{\pi}, \quad \lim_{z\to-\frac{\pi}{2}}\frac{\cos z}{z^2-\left(\frac{\pi}{2}\right)^2} = \frac{1}{\pi} \text{ similarly.}$$

8. When $z \neq z_0$, $$g(z) = \frac{1}{(z-z_0)^{m+1}}\sum_{n=m+1}^{\infty}\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^n$$
$$= \sum_{n=0}^{\infty}\frac{f^{(n+m+1)}(z_0)}{(n+m+1)!}(z-z_0)^n$$
So $z=z_0$ is a removable singularity.
$$\lim_{z\to z_0} g(z) = \lim_{z\to z_0}\sum_{n=0}^{\infty}\frac{f^{(n+m+1)}(z_0)}{(n+m+1)!}(z-z_0)^n = \frac{f^{(m+1)}(z_0)}{(m+1)!} = g(z_0)$$
So $g$ is analytic at $z=z_0$.

(c). $\int_C g(z) S_2(z)\, dz = \sum_{n=0}^{N-1} b_n \int_C \frac{g(z)}{(z-z_0)^n}\, dz + \int_C g(z) \rho_N(z)\, dz,$

where $\rho_N(z) = S_2(z) - \sum_{n=0}^{N} \frac{b_n}{(z-z_0)^n}$

Note that the convergence of the series is uniform, so that we can choose $N_0 \in \mathbb{N}$ s.t.

$$|\rho_N(z)| < \varepsilon \quad \forall N_0 \le N$$

$$\left|\int_C g(z)\rho_N(z)\, dz\right| \le \int_C |g(z)||\rho_N(z)|\, dz < ML\varepsilon,$$

where $M = \max_C |g(z)|$, $L =$ length of $C$.

So $\int_C g(z) S_2(z)\, dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} b_n \int_C \frac{g(z)}{(z-z_0)^n} dz + \lim_{N\to\infty} \int_C g(z)\rho_N(z) dz$

$$= \sum_{n=0}^{\infty} b_n \int_C \frac{g(z)}{(z-z_0)^n}\, dz$$

The remaining result follows trivially