Suggested Solutions to Quiz 3 of MATH3270A

November, 2015

yxiao@math.cuhk.edu.hk

(*15 points*)**1. Find the Laplace transform of**

$$f(t) = e^{at}\cos bt$$

where a, b **are real.**

Solution:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty e^{-st} e^{at} \frac{1}{2}(e^{ibt} + e^{-ibt})dt \\
&= \frac{1}{2}\int_0^\infty e^{(a-s+in)t}dt + \frac{1}{2}\int_0^\infty e^{(a-s-ib)t}dt \\
&= -\frac{1}{2}\frac{1}{a-s+ib} - \frac{1}{2}\frac{1}{a-s-ib} \\
&= -\frac{a-s}{(a-s)^2+b^2}
\end{aligned}$$

❏

(*15 points*)**2. Find the inverse Laplace transform of**

$$F(s) = \frac{3s}{s^2 - s - 6}.$$

Solution:

$$\frac{3s}{s^2 - s - 6} = \frac{3}{5}\left(\frac{3}{s-3} + \frac{2}{s+2}\right)$$

Since $\mathcal{L}^{-1}\{\frac{1}{s+\alpha}\} = e^{-\alpha t} \cdot u(t)$ where $u(t)$ is the Heaviside step function:

$$u(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 1 \end{cases}$$

and $\mathcal{L}^{-1}\{af(t) + bg(t)\} = a\mathcal{L}^{-1}\{f\} + b\mathcal{L}^{-1}\{g\}$.

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{3s}{s^2 - s - 6}\} &= -\frac{3}{5}\mathcal{L}^{-1}\{\frac{-3}{s-3}\} + \frac{3}{5}\mathcal{L}^{-1}\{\frac{2}{s+2}\} \\ &= \frac{9}{5}e^{3t}u(t) + \frac{6}{5}e^{-2t}u(t)\end{aligned}$$

❏

(*20 points*)**3. Find the Laplace transform of**

$$f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$$

Solution:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^1 e^{-st} \cdot 0 dt + \int_1^\infty e^{-st}(t^2 - 2t + 2) dt \\
&= \frac{1}{-s} \int_1^\infty (t^2 - 2t + 2) de^{-st} \\
&= -\frac{1}{s}(t^2 - 2t + 2)e^{-st}|_{t=1}^{t=\infty} + \frac{1}{s} \int_1^\infty e^{-st}(2t - 2) dt \\
&= \frac{1}{s} e^{-s} - \frac{2}{s^2} \int_1^\infty (t-1) de^{-st} \\
&= \frac{1}{s} e^{-s} - \frac{2}{s^2}(t-1)e^{-st}|_{t=1}^{t=\infty} + \frac{2}{s^2} \int_1^\infty e^{-st} d(t-1) \\
&= \frac{1}{s} e^{-s} - \frac{2}{s^3} e^{-st}|_{t=1}^{t=\infty} \\
&= (\frac{1}{s} + \frac{2}{s^3}) e^{-s}
\end{aligned}$$

❏

(*30 points*)**4. Find the solution of the given initial value problem**

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0$$

Solution: Apply the Laplace transform to the equation, suppose $F(s) = \mathcal{L}\{y(t)\}$. And we have:

$$\begin{aligned}
&\mathcal{L}\{y''(t)\} = s^2F(s) - sy(0) - y'(0) = s^2F(s),\\
&\mathcal{L}\{y'(t)\} = sF(s) - y(0) = sF(s),\\
&\mathcal{L}\{u(t-\tau)\} = \frac{1}{s}e^{-\tau s},\\
&\mathcal{L}\{e^{-\alpha(t-\tau)}u(t-\tau)\} = \frac{1}{s+\alpha}e^{-\tau s},
\end{aligned}$$

where $u(t)$ is the Heaviside function. Hence the equation turns into

$$s^2F(s) + 4F(s) = e^{-\pi s} - e^{-2\pi s}.$$

Thus we have:

$$\begin{aligned}
F(s) =& \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4}\\
=& \frac{1}{4i}\left(\frac{e^{-\pi s}}{s-2i} - \frac{e^{-\pi s}}{s+2i} - \frac{e^{-2\pi s}}{s-2i} + \frac{e^{-2\pi s}}{s+2i}\right).
\end{aligned}$$

So the solution is:

$$\begin{aligned}
y(t) =& \mathcal{L}^{-1}\{F(s)\}\\
=& \frac{1}{4i}\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s-2i}\} - \frac{1}{4i}\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s+2i}\} - \frac{1}{4i}\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s-2i}\} + \frac{1}{4i}\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s+2i}\}\\
=& \frac{1}{4i}u(t-\pi)e^{2i(t-\pi)} - \frac{1}{4i}u(t-2\pi)e^{2i(t-2\pi)} - \frac{1}{4i}u(t-\pi)e^{-2i(t-\pi)} + \frac{1}{4i}u(t-2\pi)e^{-2i(t-2\pi)}\\
=& \frac{1}{2}\sin 2t(u(t-\pi) - u(t-2\pi))\\
=& \frac{1}{2}\chi_{\{\pi<t<2\pi\}}\sin 2t.
\end{aligned}$$

❏

(*20 points*)**5. Find the Laplace transform of**

$$f(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$$

Solution: Note that

$$\mathcal{L}\{(h*g)(t)\} = \mathcal{L}\{h(t)\} \cdot \mathcal{L}\{g(t)\}.$$

In this case:

$$\begin{aligned}\mathcal{L}\{f(t)\} =& \mathcal{L}\{(\sin * \cos)(t)\} \\ =& \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{\cos t\}\end{aligned}$$

Direct computation:

$$\begin{aligned}\mathcal{L}\{\sin t\} &= \int_0^\infty \sin t e^{-st} dt \\ &= \int_0^\infty \frac{1}{2i}(e^{it} - e^{-it})e^{-st} dt \\ &= \frac{1}{2i}\int_0^\infty e^{(i-s)t} dt - \frac{1}{2i}\int_0^\infty e^{-(i+s)t} dt \\ &= -\frac{1}{2i}\frac{1}{i-s} - \frac{1}{2i}\frac{1}{i+s} \\ &= \frac{1}{s^2+1}.\end{aligned}$$

$$\begin{aligned}\mathcal{L}\{\cos t\} &= \int_0^\infty \cos t e^{-st} dt \\ &= \int_0^\infty \frac{1}{2}(e^{it} + e^{-it})e^{-st} dt \\ &= \frac{1}{2}\int_0^\infty e^{(i-s)t} dt + \frac{1}{2}\int_0^\infty e^{-(i+s)t} dt \\ &= -\frac{1}{2}\frac{1}{i-s} + \frac{1}{2}\frac{1}{i+s} \\ &= \frac{s}{s^2+1}.\end{aligned}$$

So the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \frac{s}{(s^2+1)^2}.$$

❏