THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH5022 Theory of Partial Differential Equations, 2nd Term 2016-17

Homework 1 (Due: March 2).

Please hand in your answers to ALL questions in class on or before the due date.

Q1. Show the Multinomial Theorem:

$$(x_1 + x_2 + \cdots x_n)^m = \sum_{|\alpha|=m} \binom{|\alpha|}{\alpha} x^\alpha,$$

where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Explain with it

$$\sum_{|\alpha|=m} \frac{1}{\alpha!} = \frac{n^m}{m!}.$$

Q2. Find the nonzero solutions in unbounded domains:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(a) $\Omega = \{x \in \mathbb{R}^n : |x| > r\}$ with $r > 0$.
(b) $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$.
Explain a little why one loses the maximum principle here.

Q3. Show the Taylor's formula with the integral remainder, i.e., for $f \in C^{m+1}(B_r(x_0))$,

$$f(x) = \sum_{|\alpha|\le m} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_m(x), \quad \forall\, x \in B_r(x_0),$$

where

$$R_m(x) = \sum_{|\alpha|=m+1} \frac{m+1}{\alpha!} \left(\int_0^1 (1-s)^m D^\alpha f(x_0 + s(x - x_0))\, ds \right) (x - x_0)^\alpha.$$

Q4. Use the mean value property to show that for a harmonic function $u \in C^1(\bar{B}_1)$,

$$\sup_{B_{1/2}} |u| \le c \left(\int_{B_1} |u|^p \right)^{1/p},$$
$$\sup_{B_{1/2}} |Du| \le c \max_{B_1} |u|,$$

where $c = c(n)$ is a constant depending only on n.

Q5. Show that a harmonic function in $\mathbb{R}^n$ with finite L^2-norm is identically zero, and also that a harmonic function in $\mathbb{R}^n$ with finite Dirichlet integral is constant.

Q6. For the problem

$$\begin{aligned} \Delta u + 2u &= 0 \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

where Ω is a rectangle domain $(0,\pi)\times(0,\pi)$ in $\mathbb{R}^2$, does one have the maximum principle? Explain your answer.

Q7. Let

$$L := a_{ij}(x)D_{ij} + b_i(x)D_i + c(x)$$

be a linear uniformly elliptic operator with coefficients in $C(\bar{\Omega})$. Apply the Hopf's maximum principle to discuss the uniqueness of solutions in $C(\bar{\Omega}) \cap C^2(\Omega)$ for the BVP

$$\begin{aligned} Lu &= f \ \text{ in } \Omega, \\ \tfrac{\partial u}{\partial n} + \alpha(x)u &= \phi \ \text{ on } \partial\Omega. \end{aligned}$$

Q8. (*Serrin's Comparison Principle*) Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $Lu \geq 0$ in Ω, where L is defined as in **Q7**. Show that if $u \leq 0$ in Ω, then either $u < 0$ in Ω or $u \equiv 0$ in Ω.

Q9. (*Varadham's Comparison Principle*) Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $Lu \geq 0$ in Ω with $u \leq 0$ on $\partial\Omega$, where L is defined as in **Q7**. Show that if the volume of Ω is small enough then $u \leq 0$ in Ω.

Q10. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy

$$\det(D^2 u) = f(x)$$

in Ω for some $f \in C(\bar{\Omega})$. Show that

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + \frac{\text{diam}(\Omega)}{|B_1|^{1/n}} \|f\|_{L^n(\Omega)}.$$