THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Fall MATH2230
Homework Set 11 (Due on)

All the homework problems are taken from Complex Variables and Applications, Ninth Edition, by James Ward Brown/Ruel V. Churchill.

P237-238: 1-4;

1 Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$; (b) $z\cos\left(\dfrac{1}{z}\right)$; (c) $\dfrac{z-\sin z}{z}$; (d) $\dfrac{\cot z}{z^4}$; (e) $\dfrac{\sinh z}{z^4(1-z^2)}$.

2 Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{e^{-z}}{z^2}$; (b) $\dfrac{e^{-z}}{(z-1)^2}$; (c) $z^2\exp\left(\dfrac{1}{z}\right)$; (d) $\dfrac{z+1}{z^2-2z}$.

3 In the example in Sec. 76. two residues were used lo evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}dz$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Sec. 77 and finding only one residue.

4 Cse the theorem in Sec. 77. involving a single residue. to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\dfrac{z^5}{1-z^3}$; (b) $\dfrac{1}{1+z^2}$; (c) $\dfrac{1}{z}$.

P246-247: 1-2, 4, 7;

1 In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z+1}{z^2+9}$; (b) $\dfrac{z^2+2}{z-1}$; (c) $\left(\dfrac{z}{2z+1}\right)^3$: (d) $\dfrac{e^z}{z^2+\pi^2}$.

2 Show that

(a) $\displaystyle\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \quad (|z| > 0,\ 0 < \arg z < 2\pi)$;

(b) $\displaystyle\operatorname*{Res}_{z=i} \frac{Log z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;

(c) $\displaystyle\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \quad (|z| > 0,\ 0 < \arg z < 2\pi)$.

4 Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz.$$

taken counterclockwise around the circle (a) $|z-2| = 2$; (b) $|z| = 4$.

7 Use the theorem in Sec. 77. involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a)$f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$; (b) $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$.

P253: 3-4;

3 Show that

(a) $\operatorname*{Res}_{z=\pi i/2} \dfrac{\sinh z}{z^2 \cosh z} = -\dfrac{4}{\pi^2}$;

(b)$\operatorname*{Res}_{z=\pi i} \dfrac{e^{zt}}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \dfrac{e^{zt}}{\sinh z} = -2\cos(\pi t)$.

4 Show that

(a)$\operatorname*{Res}_{z=z_n} z \sec z = (-)^{n+1} z_n$ where $z_n = \pi/2 + n\pi$ $(n = 0, \pm 1, \pm 2 ...)$;

(b)$\operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $z_n = (\pi/2 + n\pi)i$ $(n = 0, \pm 1, \pm 2 ...)$;.

P254: 7-8;

7 Show that

$$\int_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}.$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^2 + 3$ are the square roots of the numbers $1 \pm \sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec. 83.

8 Consider the function

$$f(z) = \frac{1}{(q(z))^2}$$

where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m = 2$ of the function f, with residue

$$B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Suggestion: Note that z_0 is a zero of order $m = 1$ of the function q, so that

$$q(z) = (z - z_0)g(z)$$

where $g(z)$ is analytic and nonzero at z_0. Then write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2} \quad \text{where} \quad \phi(z) = \frac{1}{[g(z)]^2}.$$

The desired form of the residue $B_0 = \phi'(z_0)$ can be obtained by showing that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$