Math 2230A, Complex Variables with Applications

1. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

$$(a)\frac{1-\cosh z}{z^3}; \quad (b)\frac{1-\exp(2z)}{z^4}; \quad (c)\frac{\exp(2z)}{(z-1)^2}.$$

2. Suppose that a function f is analytic at z_0, and write $g(z) = \frac{f(z)}{(z-z_0)}$. Show that

   (a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

   (b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

   Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

3. In each case, find the order m of the pole and the corresponding residue B at the singularity z=0:

$$(a) f(z) = \frac{\sinh z}{z^4}; \quad (b) f(z) = \frac{1}{z\,(e^z-1)}.$$

4. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)\,(z^2+9)} dz,$$

   taken counterclockwise around the circle (a)$|z-2| = 2$;(b)$|z| = 4$.

5. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

   taken counterclockwise around the circle (a)$|z| = 2$;(b)$|z+2| = 3$.

6. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z\,(z^2+1)} dz$$

   when C is the circle $|z| = 2$, described in the positive sense.

7. Show that

(a) $\operatorname*{Res}_{z=\pi i/2} \frac{\sinh z}{z^2 \cosh z} = -\frac{4}{\pi^2}$;

(b) $\operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

8. Show that

   (a) $\operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $z_n = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

   (b) $\operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $z_n = (\frac{\pi}{2} + n\pi)i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

9. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

$$(a) \int_C \tan z dz; \qquad \text{(b)} \int_C \frac{dz}{\sinh 2z}$$

10. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm \left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity(Exercise 8, Sec. 47), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$