Math 2230,Complex Variables with Applications

Due on Oct. 8

1. Evaluate the following integrals:
   (a) $\int_0^1 (1+it)^2 dt$;
   (b) $\int_1^2 (\frac{1}{t} - i)^2 dt$;
   (c) $\int_0^{\frac{\pi}{6}} e^{i2t} dt$;
   (d) $\int_0^{\infty} e^{-zt} dt \quad (Rez > 0)$.
2. Show that if m and n are integers,
$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when m}\neq\text{n.} \\ 2\pi & \text{when m=n.} \end{cases}$$
3. According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variavble,
$$\int_0^{\pi} e^{(1+i)x} dx = \int_0^{\pi} e^x \cos x dx + i \int_0^{\pi} e^x \sin x dx.$$
   Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.
4. f(z) is the principal branch
$$z^i = \exp(i\text{Logz}) \quad (|z| > 0, -\pi < Argz < \pi)$$
   of the power function z^i, and C is the semicircle $z = e^{i\theta} (0 \leq \theta \leq \pi)$.
5. f(z) is the principal branch
$$z^{-1-2i} = \exp[(-1-2i)\text{Logz}] \quad (|z| > 0, -\pi < Argz < \pi)$$
   of the indicated power function, and C is the contour
$$z = e^{i\theta} \quad (0 \leq \theta \leq \frac{\pi}{2}).$$
6. f(z) is the principal branch
$$z^{a-1} = \exp[(a-1)\text{Logz}] \quad (|z| > 0, -\pi < Argz < \pi)$$
   of the power function z^{a-1}, where a is a nonzero real number, and C is the positively oriented circle of radius R about the origin.

7. Let C denote the semicircular path shown in Fig.46. Evaluate the integral of the function $f(z) = \bar{z}$ along C using the parametric representation (see Exercise 2, Sec. 43)

$$(a) z = 2e^{i\theta} \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}); \quad (b) z = \sqrt{4 - y^2} + iy \quad (-2 \leq y \leq 2).$$

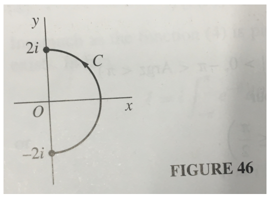

FIGURE 46

8. Let C_R denote the upper half of the circle $|z| = R(R > 2)$, taken in the counterclockwise direction. Show that

$$|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.(Compare with Example 2 in Sec. 47.)

9. Let C_R be the circle $|z| = R(R > 1)$, decribed in the counterclockwise direction. Show that

$$|\int_{C_R} \frac{\text{Log z}}{z^2} dz| < 2\pi(\frac{\pi + \ln R}{R}),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

10. Let C_ρ denote a circle $|z| = \rho(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that f(z) is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, independent of ρ, such that

$$|\int_{C_\rho} z^{-1/2} f(z) dz| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.
Suggestion: Note that since f(z) is analytic, and therefore continues, throughout the disk $|z| \leq 1$, it is bounded there (Sec. 18).