Math 2230A, Complex Variables with Applications

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if
   (a) $f(z)=\overline{z}$; (b) $f(z)=z-\overline{z}$
   (c) $f(z)=2x+ixy^2$; (d) $f(z)=e^x e^{-iy}$

2. Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when
   (a) $f(z)=iz+2$; (b) $f(z)=e^{-x}e^{-iy}$
   (c) $f(z)=z^3$; (d) $f(z)=\cos x\cosh y - i\sin x\sinh y$

3. From results obtained in Secs. 21 and 23, determine where $f'(z)$ exists and find its value when
   (a) $f(z)=1/z$; (b) $f(z)=x^2+iy^2$; (c) $f(z)=z\operatorname{Im} z$

4. (a) With the aid of the polar form (6),Sec.24, of the Cauchy-Riemann equations, derive the alternative form

   $$f'(z_0)=\frac{-i}{z_0}(u_\theta+iv_\theta)$$

   of the expression for $f'(z_0)$ found in Exercise 6.

   (b) Use the expression for $f'(z_0)$ in part (a) to show that the derivative of the function $f(z)=1/z(z\neq 0)$ in exercise 3(a) is $f'(z)=-1/z^2$.

5. Apply the theorem in Sec. 23 to verify that each of these functions is entire:
   (a) $f(z)=3x+y+i(3y-x)$; (b) $f(z)=\cosh x\cos y+i\sinh x\sin y$
   (c) $f(z)=e^{-y}\sin x-ie^{-y}\cos x$; (d) $f(z)=(z^2-2)\,e^{-x}e^{-iy}$

6. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

   (a) $f(z)=xy+iy$; (b) $f(z)=2xy+i\left(x^2-y^2\right)$; (c) $f(z)=e^y e^{ix}$

7. Let a function f be analytic everywhere in a domain D. Prove that if f(z) is real-valued for all z in D, then f(z) must be constant throughout D.

8. Show that

   $(a)\exp(2\pm 3\pi i)=-e^2$; $(b)\exp\left(\frac{2+\pi i}{4}\right)=\sqrt{\frac{e}{2}}(1+i)$
   $(c)\exp(z+\pi i)=-\exp z$

9. State why the function $f(z)=2z^2-3-ze^z+e^{-z}$ is entire.

10. Use the Cauchy-Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp\overline{z}$ is not analytic anywhere.

11. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

12. (a) Show that if e^z is real, then $\operatorname{Im} z = n\pi (n = 0, \pm 1, \pm 2, \ldots)$.
    (b) If e^z is pure imaginary, what restriction is placed on z?

13. Show that
    (a) $\log(-ei) = 1 - \frac{\pi}{2}i$; (b) $\log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

14. Show that
    (a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
    (b) $\log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
    (c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

15. (a) Show that the two square roots of i are

$$e^{i\pi/4} \quad \text{and} \quad e^{i5\pi/4}.$$

Then show that

$$\log(e^{i\pi/4}) = \left(2n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \left[(2n+1) + \frac{1}{4}\right]\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Conclude that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Show that

$$\log(i^{1/2}) = \frac{1}{2}\log i,$$

as stated in Example 5, Sec. 32, by finding the values on the right-hand side of this equation and then comparing them with the final result in part (a).

16. Show that
    (a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;
    (b) $\frac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n = 0, \pm 1, \pm 2, \ldots)$.

17. Find the principal value of
   (a) $(-i)^i$; (b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$ (c) $(1-i)^{4i}$

18. Use definition (1), Sec. 35, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

19. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 10 is the same as the principal value of $z_0^{1/n}$ defined by equation (3), Sec. 35.

20. With the aid of expression (14), Sec. 37, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots).$$