MATH 4030 Differential Geometry
Homework 9
due 10/11/2015 (Tue) at 5PM

Problems

You can directly quote results from previous Homeworks.

1. Let $c(t) : (a,b) \to \Sigma \subset \mathbb{R}^3$ be a regular parametrized curve lying on an embedded submanifold Σ. Denote $\mathcal{C} = [c]$ as the equivalence class of c up to orientation preserving reparametrizations. Recall that $c(t)$ is a geodesic if $\nabla_{c'(t)} c'(t) = 0$ for all $t \in (a,b)$, and $\mathcal{C}$ is a geodesic if the arc length parametrized $\tilde{c}(s) \in \mathcal{C}$ has geodesic curvature $k_g := \|\tilde{c}''(s)^T\| = 0$ for all s. Prove the following statements:

   (a) If $c(t)$ is a geodesic, then $\mathcal{C} = [c]$ is a geodesic.

   (b) A regular curve $\mathcal{C}$ is a geodesic if the arc length parametrized representative $\tilde{c}(s) \in \mathcal{C}$ is a geodesic.

2. Consider the abstract surface (U, g_{ij}) with $U = (0,\infty) \times (0,\pi) \subset \mathbb{R}^2$ with coordinates r, θ and

$$(g_{ij}) = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

   (a) Describe all the geodesics in (U, g_{ij}) which are complete (i.e. that can be extended indefinitely).

   (b) Calculate the Christoffel symbols Γ_{ij}^k and the Gauss curvature K.

   (c) Show that (U, g_{ij}) is isometric to the upper half plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the flat Euclidean metric.

3. (a) Recall the *Poincaré upper half-plane* $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the metric

$$(g_{ij}) = \frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Calculate the Christoffel symbols Γ_{ij}^k and the Gauss curvature K.

   (b) Consider the *Poincaré disk model* $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ equipped with the metric

$$(g_{ij}) = \frac{4}{(1-(x^2+y^2))^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Calculate the Christoffel symbols Γ_{ij}^k and the Gauss curvature K.

   (c) Show that the two abstract surfaces in (a) and (b) are isometric (*Hint: Find a fractional linear transformation that takes the upper half plane $\mathbb{R}^2_+$ to the unit disk D.*)

(d) Describe all the geodesics in each case. Is there any geodesic which is not complete?

4. (Kühnel Ch.4 Q.16, 17) Is there a regular parametrized surface $f = f(u, v) : U \to \mathbb{R}^3$ whose first and second fundamental forms are given by:

   (a) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$?

   (b) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$?

Suggested Exercises

1. (Kühnel Ch. 4 Q.23) Prove that the equations of Gauss and Codazzi are equivalent to the following two equations:

   (a) $R_{ijk\ell} := g_{is} R^s_{jk\ell} = h_{ik} h_{j\ell} - h_{i\ell} h_{jk}$.

   (b) $\nabla_i h^j_k = \nabla_k h^j_i$.

   Here $\nabla_i h^j_k$ denotes the j-th component of the tangential vector (S is the shape operator)

$$(\nabla_{\partial_i} S)(\partial_k) := \nabla_{\partial_i}(S(\partial_k)) - S(\nabla_{\partial_i} \partial_k),$$

   for a local coordinate vector fields ∂_i. As a consequence, we obtain once again the Theorema Egregium in the form

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{R_{1212}}{\det(g_{ij})}.$$

2. (Kühnel Ch.4 Q.14, 15) Let $\lambda(x)$ be a positive smooth function. For an abstract surface $(\mathbb{R}^2, g_{ij})$ with the *warped product metric*:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2(x) \end{pmatrix},$$

   (a) Calculate the Christoffel symbols Γ^k_{ij} and show that the x-lines are geodesics parametrized by arc length. What do the rest of the geodesics look like?

   (b) Determine all functions λ such that the Gauss curvature $K \equiv -1$.