MATH 4030 Differential Geometry
Homework 8
due 3/11/2015 (Tue) at 5PM

Problems

You can directly quote results from previous Homeworks. Einstein summation convention is used throughout the assignment.

1. Recall the *Poincaré upper half-plane* $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the metric

$$(g_{ij}) = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Identify the upper half plane $\mathbb{R}^2_+$ with the upper half of the complex plane $\mathbb{C}_+ := \{z = x + iy \in \mathbb{C} : y > 0\}$. Let $a, b, c, d \in \mathbb{R}$ be real constants such that $ad - bc > 0$. Let $L : \mathbb{C} \to \mathbb{C}$ be the linear fractional transformation defined by

$$L(z) := \frac{az+b}{cz+d}. \tag{1}$$

   (a) Show that L preserve the upper half plane, i.e. $L(\mathbb{R}^2_+) = \mathbb{R}^2_+$.

   (b) Prove that L is an isometry on $(\mathbb{R}^2_+, g)$, i.e. for any $p \in \mathbb{R}^2_+$ and any $v, w \in T_p\mathbb{R}^2_+$,

$$g_{L(p)}(dL_p(v), dL_p(w)) = g_p(v, w).$$

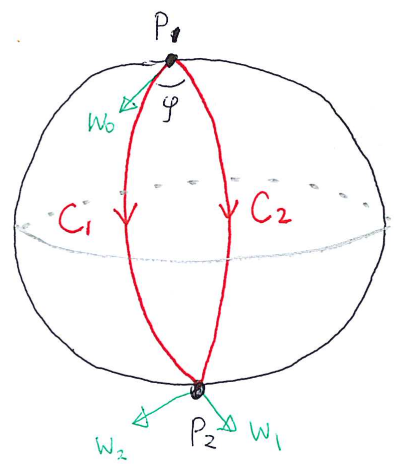
2. (do Carmo P.260 Q.4) Let $c(t)$ be a curve on a regular parametrized surface. Suppose $X(t), Y(t)$ are two tangential vector fields defined along the curve c. Prove that

$$\frac{d}{dt}g(X(t), Y(t)) = g(\nabla_{c'(t)}X(t), Y(t)) + g(X(t), \nabla_{c'(t)}Y(t)).$$

Using this result, prove that the angle between two parallel vector fields X, Y along a curve is always constant.

3. (do Carmo P. 261 Q.8) Show that if all the geodesics of a connected surface (embedded submanifold) are plane curves, then the surface is contained in a plane or a sphere.

4. (do Carmo P.261 Q.9) Consider two meridians of a sphere C_1 and C_2 which make an angle φ at the point p_1. Take the parallel transport of the tangent vector w_0 of C_1 along C_1 and C_2, from the initial point p_1 to the point p_2 where the two meridians meet again, obtaining respectively, w_1 and w_2. Compute the angle from w_1 to w_2.



5. (do Carmo P. 262 Q.15) Let p_0 be a pole of a unit sphere $\mathbb{S}^2$ and q, r be two points on the corresponding equator in such a way that the meridians p_0q and p_0r make an angle θ at p_0. Consider a unit vector v tangent to the meridian p_0q at p_0, and take the parallel transport of v along the closed curve made up by the meridian p_0q, the parallel qr, and the meridian rp_0 (Fig. 4-21).

   (a) Determine the angle of the final position of v with v.

   (b) Do the same thing when the points r, q instead of being on the equator are taken on a parallel of colatitude φ. (Note that the notation here is different from that used in our lectures $(\varphi, \theta) \leftrightarrow (\theta, \varphi)$).

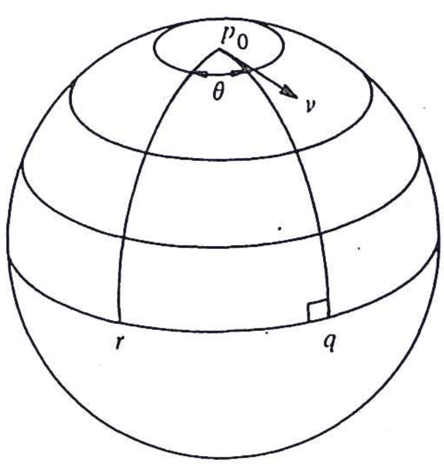


Figure 4-21

Suggested Exercises

1. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$.

   (a) Expand the tensor product $(3e_1 + 2e_2) \otimes (-e_1 + 4e_2)$.

   (b) Calculate $(3e_1 + 2e_2) \wedge (-e_1 + 4e_2)$.

   (c) Can you rewrite the tensor product $e_1 \otimes e_2 + e_2 \otimes e_1$ as $v \otimes w$ for some $v, w \in \mathbb{R}^2$?

   (d) Prove that $v \wedge w = \det(v, w) e_1 \wedge e_2$.

   (e) Compute the contraction $\iota_v(\eta \wedge \varphi)$ where $v = -2e_1 + e_2$, $\eta = 3e_1^* - 4e_2^*$ and $\varphi = e_1^* + 7e_2^*$. Here $\{e_1^*, e_2^*\}$ is the dual basis of $\{e_1, e_2\}$.

2. Consider the following vector field $X : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ defined by

$$X(x, y) := (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}) =: (F, G).$$

   (a) Show that $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$ on $\mathbb{R}^2 \setminus \{0\}$.

   (b) Prove that $X \neq \nabla f$ for *any* smooth function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$.